UNIT 14

Mathematics everywhere

Introduction

This final unit looks back at some of the main ideas that you have met in the module, and explores some new applications to illustrate that mathematics is indeed everywhere! The unit is divided into four sections, each corresponding to one of the main themes of the module.

Communicating clearly

Throughout the module you have been encouraged to explain your ideas clearly, and use graphs and diagrams appropriately. However, sometimes you have to engage with mathematical ideas that are not presented *to you* as clearly as they might have been. Section 1 considers how you can make sense of some of the numbers that you see in the media, particularly headlines about the risks of developing conditions such as cancer. This section also asks you to review your own progress in writing mathematically and highlights the importance of expressing ideas in a way that will be understood.

Extending your mathematical skills

Section 2 reviews some of the skills that you have developed in MU123, and shows you how these skills can be even more powerful when they are linked together and applied in new contexts. You'll see how the skills that you've developed in solving linear and quadratic equations can be adapted to enable you to solve simultaneous equations more complicated than those in Unit 7, and how your knowledge of trigonometry can be extended to enable you to solve some trigonometric equations.

Figure 1 A geometric puzzle from Japan

Abstract mathematics

A main theme of the module has been how mathematics can be used to describe abstract ideas and prove results. For example, in Units 5 and 9 you saw how algebra can be used to prove various results about numerical patterns and puzzles, and in Unit 8 you saw how some properties of shapes can be proved. Section 3 starts by looking at some Japanese puzzles from the eighteenth and nineteenth centuries (an example is shown in Figure 1) and then considers some general strategies for proving results involving number patterns. The final subsection illustrates how some quite abstract ideas, such as origami (the Japanese art of paper-folding), can have useful applications.

Using mathematics practically

Another main theme of the module has been how mathematical models are developed and used to solve problems in the real world. Examples include the route-planning model in Unit 2, the linear models for journeys in Units 6 and 7, the quadratic models describing the motion of projectiles in Unit 10, and the exponential models describing growth and decay in Unit 13. Section 4 shows you how you can use trigonometry to model the motion of a ferris wheel, and how to create a simple model to predict the height of the tide at different times on one particular day.

1 Communicating clearly

1.1 Interpreting information about risk

Thinking critically about numerical information that you see in the media is important if you want to make informed decisions.

You have seen other media examples in Units 1 and 11.

For example, the following headline and short extract are from a newspaper article published in 2008.

> **Heavy mobile phone use a cancer risk**
>
> People who use a mobile phone for hours a day are 50 per cent more likely to develop mouth cancer than those who do not talk on them at all, new research has shown.
>
> *The Daily Telegraph*, 18 February 2008

Figure 2 It was estimated that there were 4.6 billion mobile phone subscribers worldwide by the end of 2009.

This is quite a frightening story – many people use mobile phones on a regular basis, 50% is a large increase and mouth cancer is a potentially fatal disease if it is not treated early. So if you have a mobile phone, should you use it less frequently, or even stop using it altogether?

Before you make a decision, it's worth looking at the article carefully, checking exactly what it is claiming, trying to establish whether the evidence supports the claim and seeing if any other research on the use of mobile phones has found similar results. The full newspaper article briefly summarised some of the main points from the research report, and also commented that other studies had not so far found a link between cancer and mobile phone use. Reading one newspaper article is unlikely to provide all the information that you might need to make a decision, but you may be able to glean sufficient detail to decide whether it is worth looking into the situation further.

In the extract, the idea of *how likely* something is to occur is mentioned. This is known as the **probability** or **chance** of the event occurring. If the event is something that you hope will not occur, then the chance that it occurs is often referred to as the **risk**. (All three of these terms mean essentially the same thing.) So before looking at the extract in detail, let's consider how probability can be measured. One way of assessing how likely an event is to occur is to find how frequently it has occurred in the past. This can be done by considering a large number of cases that *could* result in the event that you are interested in, and then counting how many of the cases *did* result in that event.

To illustrate this process, suppose that you are interested in the chance of getting a girl if you or your partner have a baby. To estimate the probability of this event, researchers have to consider a large number of births and then count how many result in a girl. The proportion of babies that are girls then gives an indication of how likely it is that a girl is born. The probability of having a girl may depend on many things, such as where the mother lives, her age, or even what she eats, so the researchers need to decide which cases they will consider.

For example, they might decide to look at women living in Britain. Researchers who have taken into account a large number of randomly chosen births in Britain have found that the chance of having a girl is about 49%. That's a chance of 49 in 100. So on average about 49 out of every 100 births in Britain are those of girls. This doesn't mean that if you

look at 100 births, then you'll find that *exactly* 49 are girls. It just means that, overall, the proportion of babies in Britain that are girls is 49%.

If the researchers had looked at a different group of births, for example in another country, then they might have got a different result.

Activity 1 *The chance of having a girl*

In one year, in a small hospital in a European country, 377 girls and 367 boys were born. What percentage of the births were girls? Can you suggest why this might *not* be an accurate estimate of the probability of having a girl in the country where the hospital is?

In a similar way, the chance of a person developing a disease can be estimated by considering how many people develop the disease over a particular period of time. For example, suppose that 2 people out of a town of 10 000 people develop a particular disease in a year. Then a rough estimate of the chance of developing the disease in this town in one year is 2 in 10 000, which is the same as 1 in 5000. If a different group of people from the town were considered, for example those people with a known risk factor such as not taking exercise, then the chance of developing the disease is likely to be different. The 1 in 5000 figure gives an overall indication of the risk, but it takes into account both people with the risk factors and people without them, so an individual with the risk factors may have a higher chance of developing the disease, and an individual without the risk factors may have a lower chance.

Probabilities can be expressed as fractions, decimals or percentages, or in the form of an 'x in y' chance. For example, a 1 in 10 chance can be expressed as a probability of $\frac{1}{10}$, 0.1 or 10%. Using percentages enables you to compare the chances of different events happening fairly easily. However, one disadvantage of giving probabilities as percentages is that it can be difficult to appreciate the probability of events that are fairly unlikely to happen. For example, suppose that there is a 0.05% chance of developing a disease over your lifetime. In order to appreciate how small this probability is, it is best to convert it to an 'x in y' chance. This can be done as follows:

$$0.05\% = \frac{0.05}{100} = \frac{5}{10\,000} = \frac{1}{2000}.$$

So the chance of developing the disease is 1 in 2000.

Activity 2 *Describing chance*

In October 2009, the US National Aeronautics and Space Administration (NASA) announced that the chance that the Apophis asteroid would collide with the Earth on 13 April 2036 had fallen. The new estimate for the risk was 0.0004%. Express this probability as an 'x in 1 million' chance.

Now that you have seen how the probability of an event can be measured, let's return to the newspaper extract and examine the information that it gives. The extract says that people who use mobile phones for hours a day are 50% more likely to develop mouth cancer than those who do not use them at all. A useful question to ask when reading about percentages is

'Percentage of what?'. The extract doesn't tell you what the risk of developing mouth cancer actually is; it just tells you how much the risk is increased if you use your mobile phone a lot. This is not enough information to enable you to make an informed decision. If the risk is very small, then if it is increased by 50%, or even doubled or tripled, it will still be very small, and you might decide that it is not worth worrying about. The headline is dramatic, but it doesn't tell you anything about the chance of developing mouth cancer if you use a mobile phone, unless you know what the actual risk of developing the disease is.

Activity 3 *Reading critically*

Read the extract (on page 181) again carefully. What extra information do you think is needed before you can start to assess how likely it is that a person who uses a mobile phone will develop mouth cancer?

You have seen that an important question to sort out is: What is the actual risk of developing the type of cancer investigated in the research? Further on in the newspaper article, more details are given about the research that was carried out:

> . . . a cancer specialist at Tel Aviv University, investigated the cases of nearly 500 people diagnosed with benign and malignant tumours of the salivary gland.

So the research was based not on people suffering from mouth cancer in general, but on a group of people who had tumours of the *salivary gland*. More specifically, as the article goes on to mention, it was based on people who had tumours of the *parotid* salivary gland (Figure 3), which is the largest of the three types of human salivary gland. Furthermore, not all the tumours were malignant.

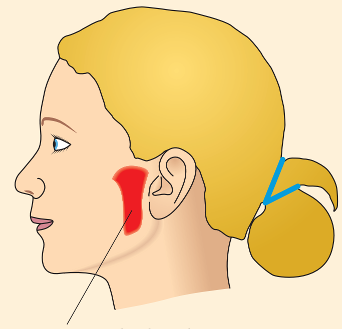


Figure 3 The parotid salivary gland

The Macmillan website is at www.macmillan.org.uk.

Malignant tumours of the salivary gland are rare. According to information published on the Macmillan Cancer Support website in 2009, at that time there were about 550 new cases of salivary gland cancer each year in the UK. The population of the UK was about 61 million in 2009, so the probability of being diagnosed with the disease in any particular year was less than 1 in 100 000. (Notice that this probability is based on all the new cases diagnosed each year, and some of these people may be heavy mobile phone users. If you considered cases in which the person had never used a mobile phone, and if using a mobile phone genuinely increases the risk, then the probability might be lower.)

Now suppose that the chance of suffering from salivary gland cancer rises by 50% among heavy mobile phone users, as suggested in the newspaper article. Then, each year in the UK, you might expect about 1.5 new cases per 100 000 heavy mobile phone users. So in a group of 200 000 heavy mobile phone users, you might expect to see 3 new cases of salivary gland cancer each year, rather than the 2 new cases that you would expect in a general group of this size. This is still a small risk.

One way to think about this risk is that in the group of 200 000 heavy mobile phone users, each year you would expect two people to develop salivary gland cancer anyway, and one extra person to develop it as a result of their heavy mobile phone use. So, if the suggestion in the newspaper article is true, then as far as developing salivary gland cancer goes, heavy use of a mobile phone will make a difference to just one person out of 200 000 each year.

The calculations on the previous page are, of course, very rough ones that do not take into account factors such as age – most people with salivary gland cancer are over the age of 55. Also, the risk referred to in the article may not have been the risk of developing the cancer in a year, but some other way of measuring the risk. For example, according to the US National Cancer Institute, the risk at birth of developing salivary gland cancer by the age of 80 is about 0.1%, or 1 in 1000. Nevertheless, the calculation does give a useful indication of the sort of risk level that you might expect if the claim in the newspaper article is true, and you use your mobile phone heavily.

If you still had doubts about using a mobile phone (or if you were investigating claims about a disease that occurs more frequently), then it would be advisable to look at the original research paper to find out how the study was carried out and if the conclusions were reported accurately. Often when articles such as the mobile phone story are published, health organisations such as the UK National Health Service (NHS) release their own assessment of the research. In this case, an NHS report stated that:

NHS reports can be found at www.nhs.uk/news.

> The '50% increased likelihood of developing mouth cancer' reported by the newspapers is mostly due to an increased risk of benign salivary gland tumours.

Benign tumours are those that do not spread or invade other organs, and they are not usually considered to be cancer. So the study did not in fact find that there was a 50% increase in the risk of developing a *malignant* tumour, which is what is normally meant by cancer.

The NHS report also noted that in the main part of the research, where regular mobile phone users (those who had made or received at least one call a week for six months) were compared with non-users, both groups were equally likely to have tumours. So although the newspaper article reported the finding of increased incidence of tumours among heavy mobile phone users, it did not report the less dramatic finding of no increased incidence among more typical mobile phone users. (However, as mentioned earlier, it did state that many other studies had found no increased risk of cancer due to mobile phone use.)

Even if a link is established between mobile phone use and salivary gland tumours, it is also important to consider whether there is a *causal* link or just a correlation, as you saw in Unit 6.

For example, suppose that smokers are more likely to use mobile phones than non-smokers, and suppose that smoking also causes salivary gland cancer. Then it is likely that there would be a higher number of cancer cases in the mobile phone users, but this may not be caused by their use of mobile phones.

So, overall the statistical facts behind the headline are not nearly so frightening as it might at first appear. Of course, there may be other health risks associated with using a mobile phone apart from the possibility of salivary gland cancer, and as further research is carried out, more convincing evidence of a link between cancer and mobile phone use may appear.

The next activity looks at a hypothetical scenario.

Activity 4 *Working out a risk*

A newspaper article claims that eating bananas regularly increases your chance of getting monkeypox sometime during your lifetime by 25%.

About 1 in 20 people who don't eat bananas get monkeypox in their lifetime.

(a) How many out of 1000 people who don't eat bananas are likely to get monkeypox sometime during their lifetime?

(b) Assume that the claim in the newspaper is true. If 1000 people eat bananas regularly, how many of these people might you expect to get monkeypox during their lifetime?

(c) If the claim in the newspaper is true, then, on average, for how many people out of 1000 does eating bananas regularly make a difference, as far as developing monkeypox goes?

Stories about risk occur frequently in the press, as they can make dramatic headlines. So if you read a headline like the one discussed in this subsection, then check the whole article (and other sources if necessary) to find out:

- exactly which disease is being discussed and which group of people is supposed to be at risk
- whether any information is given on the *actual* risk as well as the increase in the risk
- if other reports on the research are available or similar studies have been undertaken elsewhere.

In this subsection, you have seen how numerical information in a headline may give a misleading impression. This highlights the importance of reading (and writing!) articles carefully so that the ideas are understood.

1.2 Interpreting graphs and charts

In the previous subsection you saw the importance of reading media articles critically. Such articles often contain graphs or charts, as these can convey some types of numerical information more clearly and concisely than text. However, just as with text, the impression that you get when you look at a graph or chart can sometimes be misleading, and so it is important to interpret these items critically as well.

Consider the data in Table 1, which shows the estimated global number of fixed main telephone lines and mobile phone subscribers in each of the years from 1997 to 2007.

Table 1 Global growth in telephone communications (1997–2007)

Phone type (millions)	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
Main telephone lines	792	838	904	975	1034	1083	1135	1204	1262	1263	1278
Mobile phone subscribers	215	318	490	738	961	1157	1417	1763	2219	2757	3305

Source: International Telecommunications Union, 2009

The table gives a lot of detailed information, but it is difficult to see how the numbers of telephone lines and mobile phone subscribers have grown over this period, or how the numbers compare. A graph or chart is more useful for communicating these aspects of the data.

See Maths Help, Module 5 for information on using and interpreting different kinds of graphs and charts.

Figure 4 (overleaf) shows a graph, a bar chart and another type of statistical chart, each of which is a helpful and appropriate representation of some or all of the data in Table 1.

Unlike the bar charts in Unit 11, the bar chart in Figure 4(b) has *two* bars for each year represented – one bar represents the number of main telephone lines, and the other represents the number of mobile phone subscribers. A bar chart in which there are two or more bars for each data item is called a **comparative bar chart**.

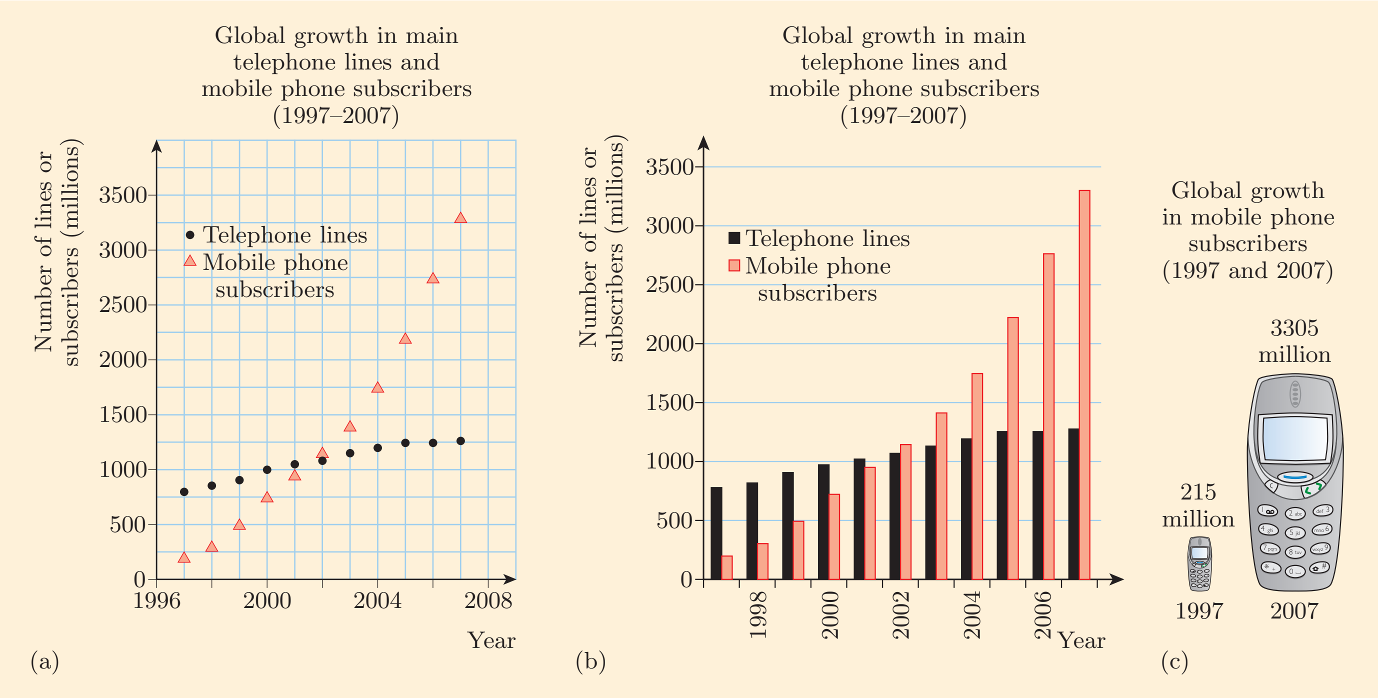


Figure 4 Three representations of data from Table 1

Notice that although the graph and bar chart in Figure 4(a) and (b) represent exactly the same data, and each does so in a way that is not misleading, they still give slightly different impressions.

The graph in Figure 4(a) makes it easy to compare the growth trends of the mobile phone data and fixed main telephone line data. To highlight the trends further, the points could be joined, or modelled by straight lines or curves, as appropriate. For example, a regression line could be fitted to the telephone line data.

On the other hand, the bar chart in Figure 4(b) makes it easy to see the *difference* between the number of telephone lines and the number of mobile subscribers in *each year* represented – these year-by-year differences are not so clear in the graph in Figure 4(a). The bar chart in Figure 4(b) also shows the long-term trends, although these are slightly more difficult to see than in the graph.

The chart in Figure 4(c) is much simpler than the graph and bar chart in Figure 4(a) and (b). It represents only the mobile phone data, not the telephone line data as well, and it depicts only the change in the number of mobile phone subscribers from 1997 to 2007, ignoring the intervening years. It uses two pictures of the front of a mobile phone and gives a visual impression that the number of subscribers in 2007 is about fifteen times larger than the number of subscribers in 1997, because approximately 15 copies of the 1997 picture would fit into the 2007 picture.

You saw another situation in which areas are used to represent numbers in Unit 11: in a histogram with unequal interval widths, the area of each bar is proportional to the number that it represents.

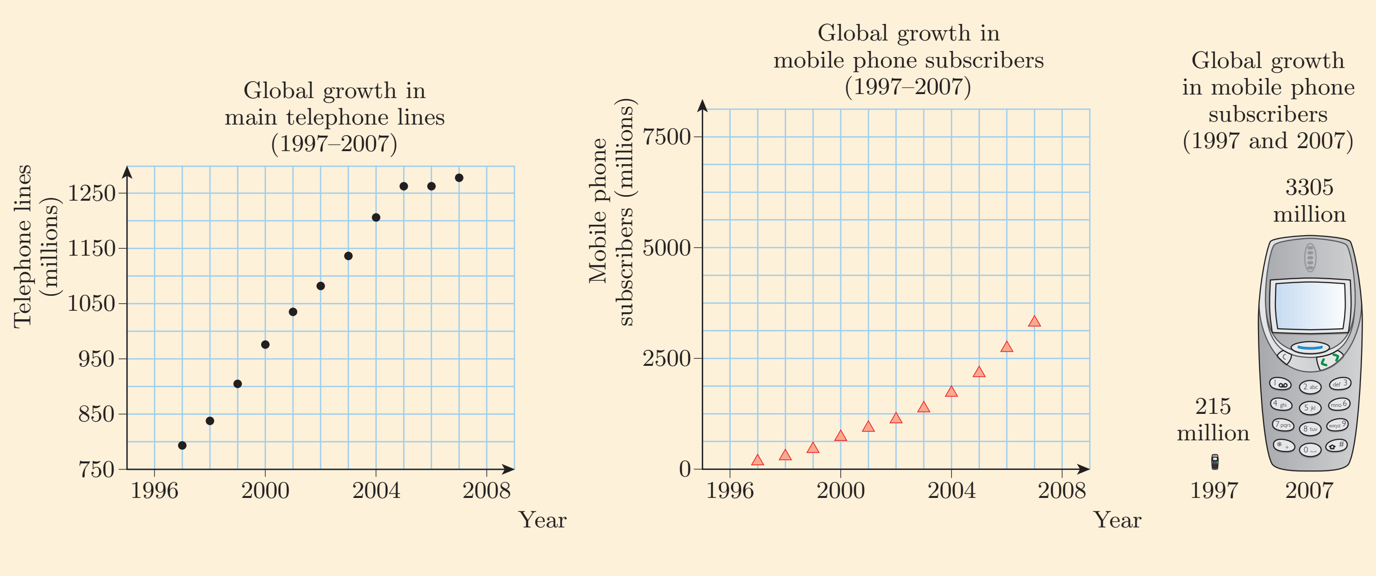
This chart is an example of a **pictogram**. The word *pictogram* describes any chart that conveys numerical information by means of pictures. Pictograms like the one in Figure 4(c) are commonly used in news media

as a way of simplifying numerical information as much as possible – the pictures represent the topic, and their relative sizes represent the numbers. In another common type of pictogram, the charts are similar to bar charts, but with the data bars made up of pictures rather than being simple rectangles. All types of pictogram can be useful for conveying statistical information quickly and clearly, especially to audiences unfamiliar with graphs and charts.

In the next activity you are asked to look at two graphs and a pictogram that also represent data from Table 1, but create different, misleading impressions.

Activity 5 *Looking at statistical charts critically*

The graphs and pictogram below represent data from Table 1.



(a) If you were to glance at the two graphs on the left together, what misleading impression might they give you about how the growth in the number of mobile phone subscribers compares to the growth in the number of fixed main telephone lines? What causes this misleading impression?

(b) In the pictogram on the right, the picture that represents the number of mobile phone subscribers in 2007 is scaled up by a scale factor of 15 from the corresponding 1997 picture, to represent the fact that there were approximately 15 times more mobile phone subscribers in 2007 than in 1997. Why is this pictogram not an appropriate way to represent this fact?

A feature of the first graph in Activity 5 that is particularly worth noticing is that the vertical scale does not start at zero. This type of scale can give you the impression that the change in a quantity is much more significant than it actually is.

Activity 5 illustrates that when you are trying to interpret graphs and charts that have been produced by someone else, it is important to think about whether the data have been represented fairly, or whether the author might have represented them in an inappropriate way, perhaps to

try to back up a point that he or she is making. You should also consider whether the data are from a reliable source and whether the author has ignored any important aspects of the data.

When you present your own mathematics, either in text or in graphs and diagrams, bear in mind that your readers might be critically assessing your work in a similar way to the examples in this section! So try to express your ideas clearly and concisely, using graphs and diagrams appropriately. Use sound logical arguments, and make sure that any use of statistical data is not misleading. You have been developing your skills in these areas throughout MU123.

In the final activity in this section you are asked to look back over your assignments, to see if there are any points that you need to consider as you tackle your final written assignment.

Activity 6 *Presenting your mathematics*

(a) Look back over your written assignments and your tutor's comments. What are the main points that you have learned about presenting your work?

(b) The final written assignment contains questions based on topics from throughout the module. If you have not already done so, download the assignment from the module website and look through the questions.

From your review in part (a), are there any points that you need to bear in mind when you tackle this assignment?

2 Extending your mathematical skills

During your study of MU123 you have developed a variety of skills for solving both abstract and practical mathematical problems. Many of these problems involve solving equations, that is, finding the values of the unknowns that satisfy the equations. This section concentrates on your equation-solving skills. You will see how the skills that you have already acquired can be extended to allow you to solve new problems, and you will learn some new skills for solving *trigonometric equations* – equations involving trigonometric functions. These skills will be needed if you plan to study more advanced mathematics modules.

2.1 Developing your equation-solving skills

Some of the equation-solving methods that you have met in MU123 are specific to particular kinds of equations. For example, you have seen how to solve quadratic equations by using the quadratic formula, and how to solve exponential equations by taking logarithms. Other methods can be used for different types of equations, or adapted for different types of equations, even though you might have met them in the context of a specific type of equation. Also, you can often combine more than one method to give you a way of solving a particular problem. You will see some examples of this in this section. Learning to adapt and combine your skills in this way is an important part of learning mathematics, as it extends the range of problems that you can deal with.

When you are choosing a method for solving an equation, or trying to adapt a method that you have seen in another context, one aspect that you have to bear in mind is whether you want an exact answer, or whether an approximate one will do. You have met various algebraic techniques that can give exact answers, such as 'doing the same thing to both sides', or using the quadratic formula. However, approximate answers are usually good enough for practical purposes, and you have also seen some techniques that give approximate answers, such as using graphs, and trial and improvement.

In this subsection you will see how you can extend both graphical and algebraic methods to new situations.

Adapting graphical methods

Suppose, for example, that you want to solve the quadratic equation

$$2x^2 + 2x = 9. \tag{1}$$

If approximate solutions are good enough, then you could use the graphical method that you saw in Unit 10 to find these. You rearrange the equation into the form

$$2x^2 + 2x - 9 = 0,$$

and then use Graphplotter to plot the graph of the associated quadratic function, $y = 2x^2 + 2x - 9$, as shown in Figure 5. The x-intercepts of the graph are the values of x for which $y = 0$, and hence are the solutions of the equation. You can find these accurately to a given number of decimal places by using a procedure given in Unit 10, which you will be reminded about shortly.

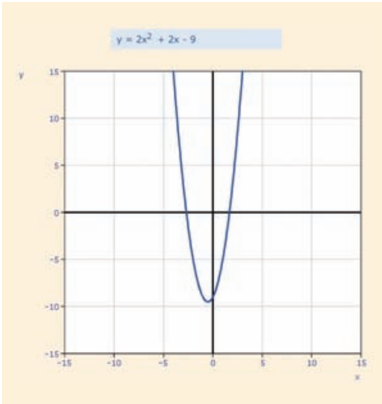


Figure 5 The graph of $y = 2x^2 + 2x - 9$

However, sometimes it is useful to solve an equation graphically by plotting, on the same axes, the two functions associated with the left- and right-hand sides of the equation, rather than by rearranging the equation and plotting a single function. For example, for equation (1) you would plot the graphs of the functions

$$y = 2x^2 + 2x \quad \text{and} \quad y = 9,$$

as shown in Figure 6. The x-coordinate of any point where the two graphs cross is a value of x for which the two y-values are the same – in other words, for which the left- and right-hand sides of equation (1) are equal. That is, it is a solution of the equation.

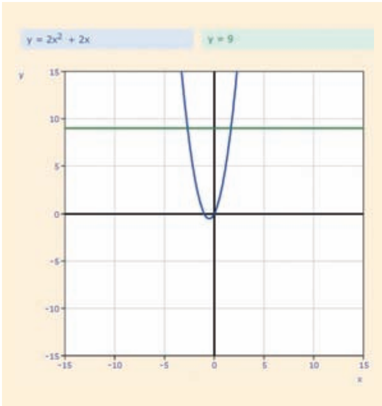


Figure 6 The graphs of $y = 2x^2 + 2x$ and $y = 9$

The usual reason why you would want to plot the two sides of an equation separately is that they represent particular quantities in the context that the equation comes from, so it is more meaningful to plot them than the single function given by the rearranged equation.

To use a Graphplotter graph of the two sides of an equation to find the solutions accurately to a given number of decimal places, you can adapt the procedure that you saw in Unit 10. Suppose that you want to use the graph in Figure 6 to find the solutions of equation (1) to two decimal places. You tick the Trace option, and zoom in on a crossing point until the x-coordinates of the trace points are shown to at least three decimal places (that is, to at least one more decimal place than the number that you eventually want). If you can find two trace points, one below the crossing point and one above, whose x-coordinates are the same when rounded to two decimal places, then the x-coordinate of the crossing point must also be the same when rounded to two decimal places.

For example, the two screenshots in Figure 7 show the left-hand crossing point in Figure 6. Figure 7(a) shows that there is a trace point below the crossing point with an x-coordinate of -2.676 to three decimal places, and Figure 7(b) shows that there is a trace point above the crossing point with an x-coordinate of -2.681 to three decimal places. Since both of these values are -2.68 to two decimal places, and the x-coordinate of the crossing point lies between these two values, it is also -2.68 to two decimal places. You can find in a similar way that the other crossing point in Figure 6 has x-coordinate 1.68. So the two solutions of equation (1) are -2.68 and 1.68, to two decimal places.

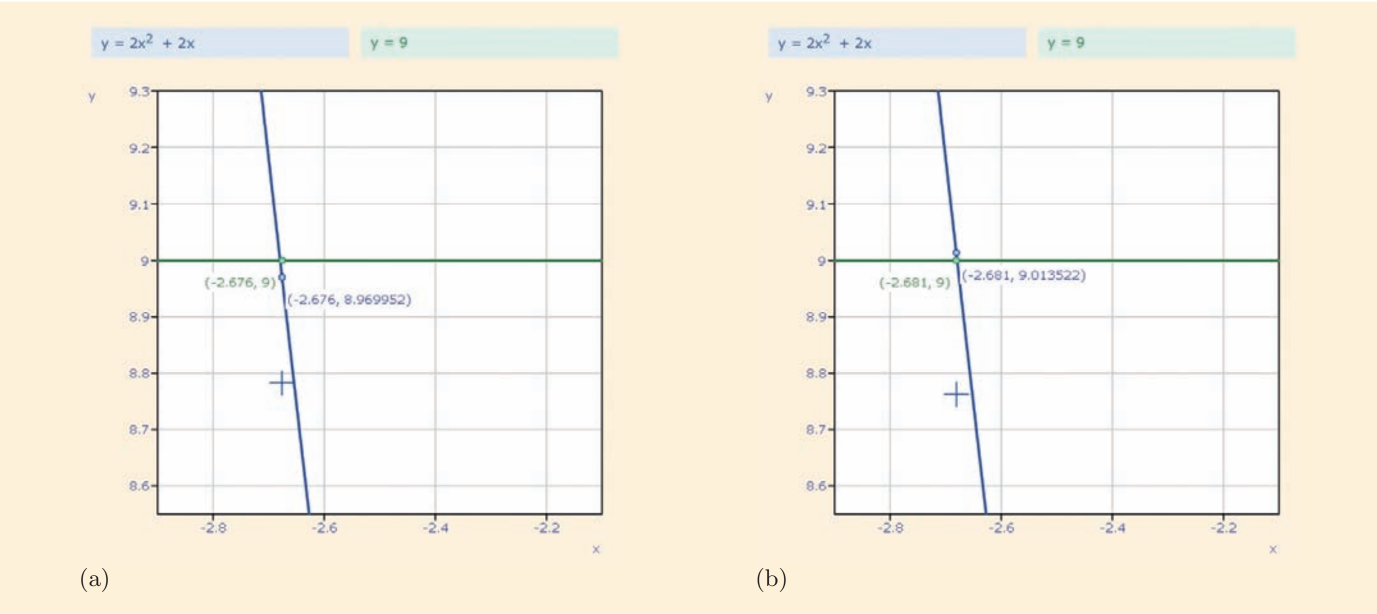


Figure 7 A crossing point of the graphs of $y = 2x^2 + 2x$ and $y = 9$, and trace points (a) below and (b) above the crossing point

You can use a similar method to find approximate values for the solutions of any equation. If you need to plot the graph of an equation whose form is not one of the standard ones in the Graphplotter drop-down list, then you can use 'Custom function', which is also available from the drop-down list. For example, to plot the graph of the equation $y = 4(x^3 - 2)$, choose Custom function and type

```
4*(x^3-2)
```

into the '$y =$' box. You can find more information about using Custom function on the Graphplotter Help page (press the orange 'Help' button at the top right of Graphplotter).

Graphplotter

Activity 7 *Solving an equation in different ways*

(a) Plot the two sides of the equation

$$(x - 3)(x + 2) = 2$$

on the same axes in Graphplotter. Then use the graphical method described on page 189 to solve the equation, giving your answers to two decimal places.

(b) Solve the same equation algebraically. Evaluate your answers to two decimal places and check that they match those in part (a).

In Unit 7 you saw that you can find an approximate solution of two simultaneous linear equations by drawing their graphs and reading off the x- and y-coordinates of the crossing point. The next activity illustrates that you can use the same method for any simultaneous equations in two unknowns, even if they are not linear. The x- and y-coordinates of the crossing point can both be found to a given number of decimal places by using the method discussed earlier in this subsection.

Activity 8 *Solving simultaneous equations graphically*

Graphplotter

Consider the following simultaneous equations, in which one is linear and one is quadratic:

$$y = 2x + 3,$$
$$y = x^2 - 2x + 1.$$

(a) Use Graphplotter to plot the graphs of the two equations on the same axes, and to find the coordinates of their crossing points to two decimal places.

(b) Explain why these coordinates are the solutions of the simultaneous equations.

Adapting algebraic methods

You can also often adapt the algebraic methods that you have met for solving equations to allow you to deal with new situations. For example, you may be able to adapt the algebraic methods for solving simultaneous linear equations that you met in Unit 7 to allow you to solve simultaneous equations in which at least one equation is not linear. Suppose that you want to solve the following simultaneous equations algebraically:

$$y = 2 - 4x,$$
$$y = 2x^2 + x - 1.$$

A good strategy when faced with a new problem like this is to ask yourself: Have I seen anything with similarities to this before, and if so, can I use a similar approach here? In this case, you need to solve a linear equation and a quadratic equation simultaneously. In Unit 7 you met methods for solving simultaneous equations in which both equations are linear, and in Units 9 and 10 you saw some methods for solving quadratic equations, such as factorisation and the quadratic formula. So putting these ideas together could help you to solve simultaneous equations in which one equation is linear and the other is quadratic. This approach is used in the example below.

Example 1 *Solving non-linear simultaneous equations algebraically*

Use algebra to solve the following simultaneous equations:

$$y = 2 - 4x,$$
$$y = 2x^2 + x - 1.$$

Solution

Try a method similar to one that you saw in Unit 7.

The equations are

$$y = 2 - 4x, \qquad (2)$$
$$y = 2x^2 + x - 1. \qquad (3)$$

The right-hand sides must be equal to each other, which gives

$$2 - 4x = 2x^2 + x - 1.$$

Simplify and solve this quadratic equation.

Simplify: $0 = 2x^2 + 5x - 3$
Factorise: $0 = (2x - 1)(x + 3)$
So: $2x - 1 = 0$ or $x + 3 = 0$
So: $x = \frac{1}{2}$ or $x = -3$

Substitute into one of the original equations to find the values of y.

Substituting $x = \frac{1}{2}$ into equation (2) gives

$$y = 2 - 4 \times \tfrac{1}{2} = 2 - 2 = 0.$$

Substituting $x = -3$ into the same equation gives

$$y = 2 - 4 \times (-3) = 2 + 12 = 14.$$

So the solutions are

$$x = \tfrac{1}{2},\ y = 0 \quad \text{and} \quad x = -3,\ y = 14.$$

(Check: Substituting $x = \frac{1}{2}$, $y = 0$ into equation (3) gives

$$\text{RHS} = 2 \times \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} - 1 = \tfrac{1}{2} + \tfrac{1}{2} - 1 = 0 = \text{LHS},$$

and substituting $x = -3$, $y = 14$ into the same equation gives

$$\text{RHS} = 2 \times (-3)^2 + (-3) - 1 = 18 - 3 - 1 = 14 = \text{LHS}.)$$

Now let's just pause to think about what has been achieved here. A pair of simultaneous equations in which one equation is linear and one is quadratic was solved using algebra, by adapting and combining methods that you have seen for solving simultaneous linear equations and for solving single quadratic equations. There could be many other situations where you need to solve two simultaneous equations in which at least one equation is not linear, so this is an important step forward. Adapting and combining techniques with which you are already familiar has opened up a new set of problems that you can solve.

Activity 9 *Solving simultaneous equations algebraically*

For further practice, you might like to try solving the simultaneous equations in Activity 8 algebraically.

Use algebra to solve the following simultaneous equations:

$$y = x + 1,$$
$$y = x^2 + 2x - 5.$$

Using shortcuts

Another way in which you can develop your equation-solving skills is that, as you become more confident, you may be able to take shortcuts that

reduce some of your working. You must be sure that you understand the reasoning behind such a shortcut, so that you are aware of the situations where the shortcut can be used and those where it cannot.

For instance, there is a shortcut that you can sometimes use when you are rearranging equations involving algebraic fractions. You saw earlier in the module that usually the first step in solving an equation involving algebraic fractions is to multiply both sides by an expression that is a multiple of all the denominators. This process clears the fractions, making the equation easier to solve.

For example, here is how you would solve the equation

$$\frac{4}{x-3} = \frac{3}{x-2}, \tag{4}$$

using the method that you have seen.

Assume that $x \neq 3$ and $x \neq 2$.

You saw in Unit 9 that when you multiply an equation by an expression you must assume that the expression is non-zero. You should check that any solutions that you obtain at the end of your working satisfy your assumption.

Multiplying by $(x-3)(x-2)$ gives

$$(x-3)(x-2)\frac{4}{(x-3)} = (x-3)(x-2)\frac{3}{(x-2)}.$$

Cancelling gives

$$4(x-2) = 3(x-3). \tag{5}$$

Hence

$$4x - 8 = 3x - 9$$

which gives $x = -1$.

This answer satisfies the initial assumptions $x \neq 3$ and $x \neq 2$, so the solution is $x = -1$.

Equation (4) is of a particular form for which there is a shortcut, known as **cross-multiplying**, that reduces some of the work in rearranging the equation. The shortcut applies to equations of the form

$$\frac{A}{B} = \frac{C}{D}, \tag{6}$$

where A, B, C and D are expressions. If you look back at equation (4), then you will see that it is of this form with $A = 4$, $B = x - 3$, $C = 3$ and $D = x - 2$.

Cross-multiplying allows you to remove the fractions in an equation of form (6) in one step instead of two. If you assume that the expressions on the denominators, B and D, are not equal to zero, and then multiply both sides of the equation by their product, BD, to clear the fractions, you obtain

$$BD \times \frac{A}{B} = BD \times \frac{C}{D}.$$

Cancelling gives

$$AD = BC.$$

You can think of this equation as being obtained from equation (6) by multiplying across the equals sign, like this:

$$\frac{A}{B} = \frac{C}{D} \quad \text{gives} \quad AD = BC.$$

This is the technique known as cross-multiplying.

For example, consider equation (4) again:

$$\frac{4}{x-3} = \frac{3}{x-2}.$$

To cross-multiply in this equation, you multiply across the equals sign like this:

$$\frac{4}{x-3} \qquad \frac{3}{x-2}.$$

This gives

$$4(x-2) = 3(x-3),$$

which is equation (5), reached in a more straightforward way. You can solve the equation in the usual way from this stage. You still need to make the assumptions $x \neq 3$ and $x \neq 2$, of course, as cross-multiplying is just a shortened form of the usual method.

Activity 10 *Using cross-multiplication to solve equations*

Which two of the following three equations are in a suitable form for cross-multiplying? Solve the three equations.

(a) $\dfrac{x}{x+1} = \dfrac{2}{x+3}$ (b) $\dfrac{x}{x+1} = 2$ (c) $\dfrac{x}{x+1} = \dfrac{2}{x} + 3$

Remember that you should use shortcuts only in situations where you are confident that they apply. If you are not sure, then stick to a basic method, to avoid the possibility of errors.

2.2 Solving trigonometric equations

In Unit 12 you saw that when you use trigonometry to find an unknown angle θ in a triangle, the final step involves solving a trigonometric equation such as

In equations of this type involving $\sin\theta$ and $\cos\theta$, there is a solution only if the number on the right-hand side is between -1 and 1, inclusive.

$$\sin\theta = \tfrac{5}{6}, \quad \cos\theta = -0.87 \quad \text{or} \quad \tan\theta = 4.$$

Equations like these crop up frequently in trigonometry. You'll see them arise in three more situations in this unit, starting in the next subsection, where they occur when you calculate the angle that a straight line makes with the x-axis. Then in Section 4 you'll see equations of this sort when the motion of a ferris wheel and the height of a tide over a day are modelled.

Remember that an *acute* angle is one between 0° and 90°, exclusive. Similarly, an *obtuse* angle is one between 90° and 180°, exclusive.

As you know, you can obtain a solution of an equation like those above by using the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ function on your calculator. If you know that the angle θ that you are trying to find is *acute*, then the solution given by your calculator will be the angle that you want. In general, however, the solution given by your calculator is only one of many possible solutions.

For example, consider the equation

$$\sin\theta = \tfrac{5}{6}.$$

One solution of this equation is

$$\theta = \sin^{-1}\left(\tfrac{5}{6}\right) \approx 56°.$$

However, the graph in Figure 8 shows that there are other solutions. Each of the points marked with a black dot corresponds to a value of θ for which

$\sin\theta = \frac{5}{6}$. The dot between 0° and 90° corresponds to the solution on the opposite page, $\theta \approx 56°$, but you can see that there is another solution between 90° and 180°, and since the graph of $y = \sin\theta$ repeats every 360°, there are infinitely many solutions altogether.

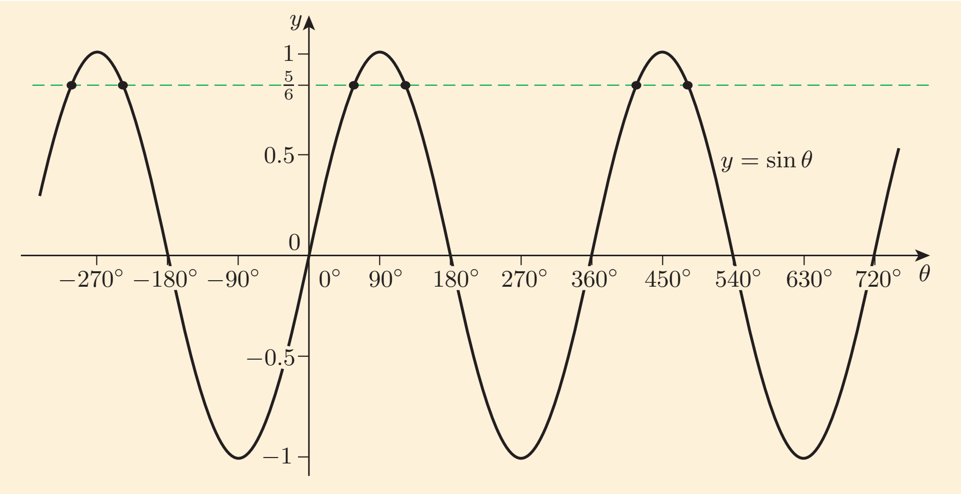


Figure 8 Some points on the graph of $y = \sin\theta$ that have y-coordinate $\frac{5}{6}$

When you need to solve a trigonometric equation like those on the opposite page, you often need solutions other than the one provided by your calculator. You saw an example of this in Unit 12. When you use the Sine Rule to find an angle θ of a triangle, you obtain an equation of the form

$$\sin\theta = \text{a number},$$

where the number on the right-hand side is positive, and less than or equal to 1. You can use the $\sin^{-1}$ function on your calculator to find one solution of the equation, but, except when $\theta = 90°$, there is always a second solution, which is 180° minus the first solution (as illustrated in Figure 8 for the equation $\sin\theta = \frac{5}{6}$). Both of the two solutions can occur as the angle θ of the triangle, and you need more information about the triangle in order to decide which is the correct angle.

You saw how to use the Sine Rule to find an angle of a triangle in Unit 12, Subsections 2.1 and 3.3.

So being able to find all the solutions of trigonometric equations is a useful skill. Once you know how to find all the solutions of an equation of this type, you can choose any particular solution that you might need for the situation that you're working with. For example, you might know that the angle that you are trying to find is between 90° and 180°.

You can use Graphplotter to find approximate solutions of trigonometric equations, just as you can for any other type of equation, and you will be asked to use this method in Section 4.

However, as you have seen for other kinds of equations, it is also useful to have a non-graphical method for finding solutions. This often allows you to obtain solutions more quickly, it can enable you to obtain exact solutions rather than approximate ones, and it gives you a greater understanding of the mathematics.

So in this subsection you will learn a useful method for finding all the solutions of trigonometric equations like those at the beginning of this subsection.

The key to finding all the solutions of trigonometric equations is to understand how the sine, cosine and tangent of any angle are related to the sine, cosine and tangent of an acute angle. So let's look at that next.

Sines, cosines and tangents of related angles

The sine, cosine and tangent of a general angle were defined in Subsection 3.1 of Unit 12.

Think back to the way that the sine, cosine and tangent of a general angle were defined. Suppose that the angle is θ: remember that you think of it drawn on a pair of coordinate axes, as shown in Figure 9. It is measured from the positive direction of the x-axis, anticlockwise if θ is positive, and clockwise if θ is negative.

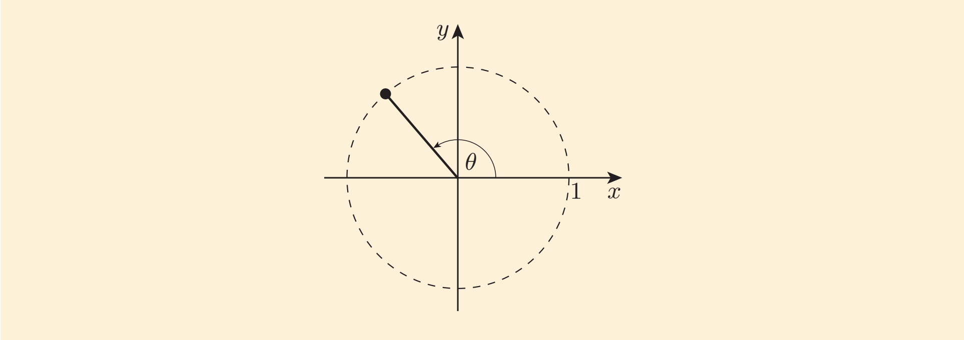


Figure 9 An angle θ drawn on a pair of coordinate axes

The angle θ corresponds to a point on the unit circle (the circle of radius 1 centred at the origin), and $\sin\theta$ and $\cos\theta$ are defined to be the y- and x-coordinates of this point, respectively. The value of $\tan\theta$ is defined by

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

For example, consider the angle 25°. Figure 10 shows the point P on the unit circle that corresponds to 25°.

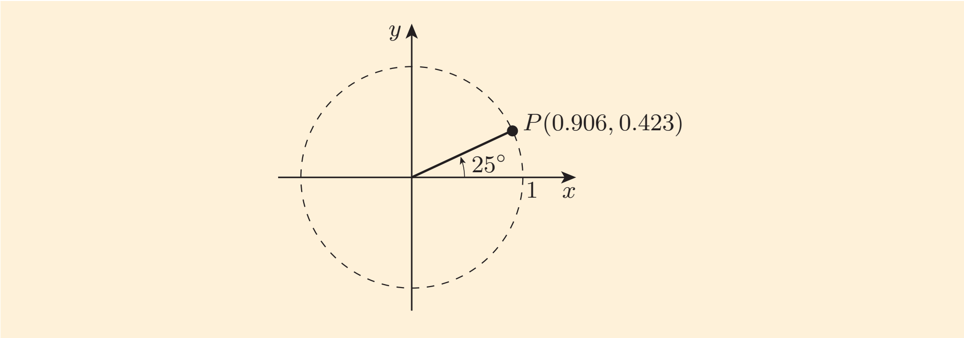


Figure 10 The point P corresponding to the angle 25°

The y-coordinate of P is 0.423, to three decimal places, so

$$\sin 25^\circ \approx 0.423.$$

The x-coordinate of P is 0.906, to three decimal places, so

$$\cos 25^\circ \approx 0.906.$$

Using more precise values for $\cos 25^\circ$ and $\sin 25^\circ$ gives

$$\tan 25^\circ = \frac{\sin 25^\circ}{\cos 25^\circ} = \frac{0.422\,61\ldots}{0.906\,30\ldots} \approx 0.466.$$

The angle 25°, shown in Figure 10, is referred to as a *first-quadrant* angle, because the corresponding point P lies in the first quadrant. Similarly, any angle that corresponds to a point on the unit circle that lies in the second quadrant is called a *second-quadrant* angle, and so on. Figure 11 reminds you of how the quadrants are labelled. So, for example, any acute angle is a first-quadrant angle, and any obtuse angle is a second-quadrant angle.

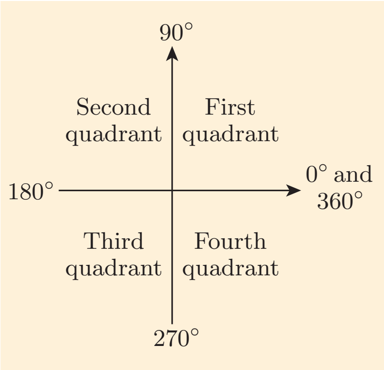


Figure 11 The four quadrants

Now consider what happens if you reflect the point P corresponding to 25° in the y-axis. The resulting point Q is shown in Figure 12. The angle corresponding to Q is a second-quadrant angle, namely $180° - 25° = 155°$. The x-coordinate of Q is the negative of the x-coordinate of P, and the y-coordinate of Q is the same as the y-coordinate of P.

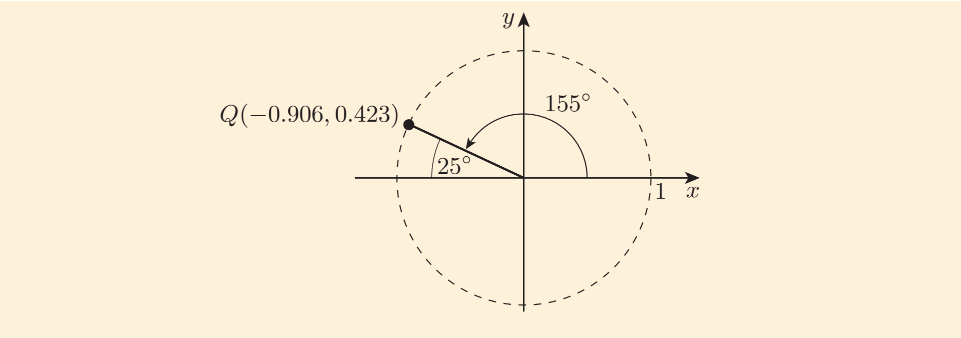


Figure 12 The point Q corresponding to the angle 155°

The sine and cosine of 155° are given by the coordinates of Q. These coordinates give:

$$\sin 155° \approx 0.423, \quad \cos 155° \approx -0.906$$

and

$$\tan 155° = \frac{0.422\,61\ldots}{-0.906\,30\ldots} \approx -0.466.$$

So the sine, cosine and tangent of 155° are exactly the same as the sine, cosine and tangent of 25°, except for some of the signs. The cosine and tangent of 155° are negative, whereas the cosine and tangent of 25° are positive. Another way to describe the relationship between the sines, cosines and tangents of these two angles is to say that they have the same *magnitudes*.

Remember that the *magnitude* of a number is the number without its negative sign, if it has one.

There are two other angles in the range 0° to 360° that also have the same sine, cosine and tangent as 25°, except for some of the signs, as shown in Figure 13.

The first of these other angles is obtained by rotating the point P through a half-turn about the origin (or alternatively by reflecting the point Q in the x-axis). This gives the point R shown in Figure 13(a).

The second of the other angles is obtained by reflecting the point P in the x-axis (or alternatively by reflecting the point R in the y-axis). This gives the point S shown in Figure 13(b).

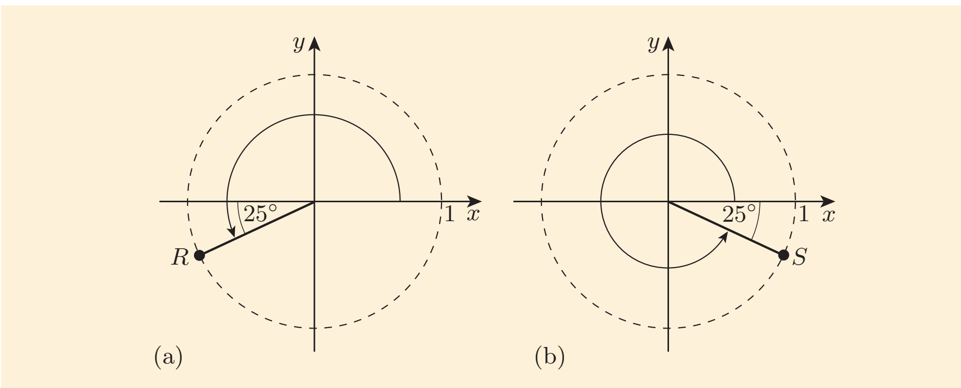


Figure 13 (a) The point R obtained by rotating P through a half-turn about the origin. (b) The point S obtained by reflecting P in the x-axis.

In the next activity you are asked to work out the angles corresponding to R and S, and find their sines, cosines and tangents.

Activity 11 *Finding the cosines, sines and tangents of angles related to* 25°

(a) Work out the angles, measured anticlockwise from the positive x-axis as shown in Figure 13, corresponding to the points R and S.

(b) By considering how the coordinates of R and S are related to the coordinates of P (or Q), write down the coordinates of R and S, to three significant figures.

(c) Using your answers to part (b), find the sine, cosine and tangent of each of the two angles in part (a), to three significant figures. (For the tangents, you will need to use more precise values of the cosines and sines, to avoid rounding errors. You can find the numbers that you need on page 196.)

You have now seen that the four angles

$$\begin{aligned}&25^\circ,\\ &180^\circ - 25^\circ = 155^\circ,\\ &180^\circ + 25^\circ = 205^\circ,\\ &360^\circ - 25^\circ = 335^\circ\end{aligned}$$

all have the same sine, cosine and tangent, except for the signs.

In general, you can see that, for any acute angle ϕ, the four angles

$$\phi, \quad 180^\circ - \phi, \quad 180^\circ + \phi \quad \text{and} \quad 360^\circ - \phi$$

all have the same sine, cosine and tangent, except for the signs. These four related angles are shown in Figure 14.

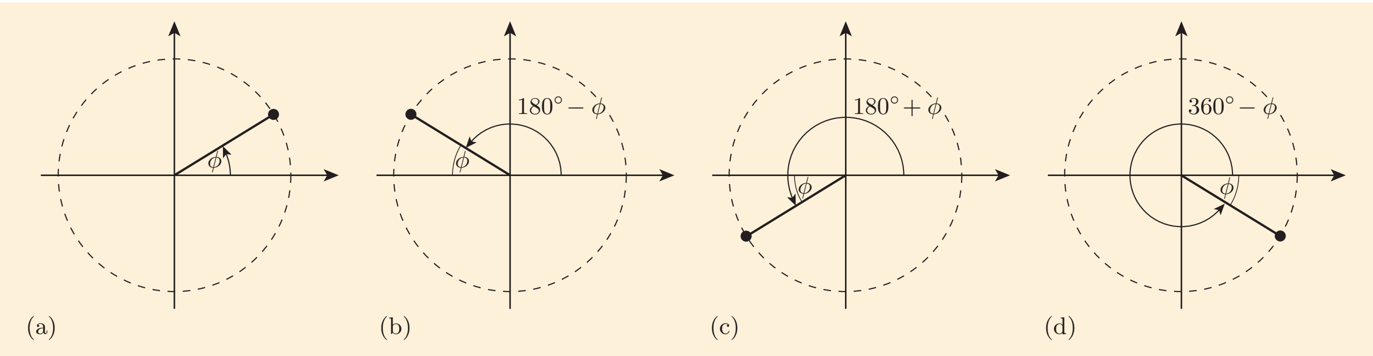


Figure 14 Four angles with the same sine, cosine and tangent, except for the signs

One of the four related angles lies in each of the four quadrants. The summary diagram in Figure 15 should help you to remember them.

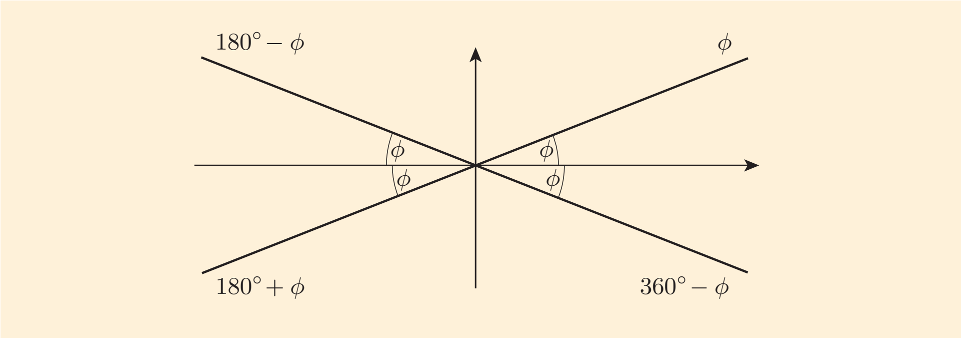


Figure 15 The related angles diagram

The signs of the sines, cosines and tangents are determined by the signs that the x- and y-coordinates take in the different quadrants, as follows.

- In the first quadrant, x and y are both positive, so sine, cosine and tangent are all positive.
- In the second quadrant, x is negative and y is positive, so sine is positive and cosine is negative, and hence tangent is negative.
- In the third quadrant, x and y are both negative, so sine and cosine are both negative, and hence tangent is positive.
- In the fourth quadrant, x is positive and y is negative, so sine is negative and cosine is positive, and hence tangent is negative.

The sign of the tangent is worked out from the signs of the sine and cosine by using the fact that

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

for any angle θ.

There is a useful way to remember these signs, which is shown in Figure 16. The letters tell you which of sine, cosine and tangent are positive in which quadrants:

A stands for all,
S stands for sine,
T stands for tangent,
C stands for cosine.

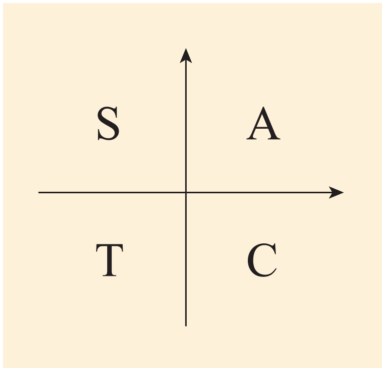


Figure 16 The CAST diagram

To remember this diagram, you might like to think of the word CAST or the mnemonic phrase ‘All Silly Tom Cats’.

Using the CAST diagram and the related angles diagram to solve trigonometric equations

You can use the information in the the CAST diagram and the related angles diagram to help you to solve simple trigonometric equations. The method is demonstrated in the example on the next page.

You usually also need to use your calculator. Whenever you use your calculator for trigonometry, remember to check that it is set to use the units for angles that you are working with – degrees or radians.

Subsections 3.7 and 3.9 of the MU123 Guide tell you how to set the calculator recommended for the module to degrees or radians.

Example 2 *Solving trigonometric equations*

Find all the solutions between 0° and 360° of the following equations. Give your answers to the nearest degree.

(a) $\cos\theta = 0.8$ (b) $\tan\theta = -4$

Solution

(a) Use the CAST diagram to find the quadrants of the solutions.

The cosine of θ is positive, so θ is a first- or fourth-quadrant angle.

Use your calculator to find the first-quadrant angle.

One solution of the equation is

$$\theta = \cos^{-1}(0.8) = 37^\circ \text{ (to the nearest degree).}$$

Use the related angles diagram to find the related fourth-quadrant angle.

The other solution is

$$\theta = 360^\circ - 37^\circ = 323^\circ \text{ (to the nearest degree).}$$

(Check: A calculator gives

$$\cos 37^\circ = 0.798\ldots \approx 0.8,$$
$$\cos 323^\circ = 0.798\ldots \approx 0.8.)$$

(b) Use the CAST diagram to find the quadrants of the solutions.

The tangent of θ is negative, so θ is a second- or fourth-quadrant angle.

Use your calculator to find the related first-quadrant angle whose tangent has the same magnitude but is positive.

The related first-quadrant angle is

$$\tan^{-1}(4) = 76^\circ \text{ (to the nearest degree).}$$

Use the related angles diagram to find the related second and fourth-quadrant angles.

The solutions are

$$\theta = 180^\circ - 76^\circ = 104^\circ \text{ (to the nearest degree),}$$
$$\theta = 360^\circ - 76^\circ = 284^\circ \text{ (to the nearest degree).}$$

(Check: A calculator gives

$$\tan 104^\circ = -4.010\ldots \approx -4,$$
$$\tan 284^\circ = -4.010\ldots \approx -4.)$$

Sometimes you might need to find the solutions of one of the seven 'special' trigonometric equations below.

$\sin\theta = 1$,
$\cos\theta = 1$,

$\sin\theta = -1$,
$\cos\theta = -1$,

$\sin\theta = 0$,
$\cos\theta = 0$,
$\tan\theta = 0$.

A good way to solve one of these equations is not to use the CAST diagram and the related angles diagram, but instead to just look at the graph of the sine, cosine or tangent function, as appropriate.

For example, you can see from the graph of the sine function (which is on page 94, and in the Handbook) that the only solution of the equation $\sin\theta = 1$ between 0° and 360° is $\theta = 90^\circ$.

Before you do the next activity, check that your calculator is set to use degrees rather than radians, if you haven't done so already.

Activity 12 *Solving trigonometric equations*

Find all the solutions between 0° and 360° of the following equations. Give your answers to the nearest degree.

(a) $\sin\theta = 0.2$ (b) $\cos\theta = -0.6$

Once you have found all the solutions of a trigonometric equation in the interval 0° to 360°, it is straightforward to find any other solutions that you want. The trigonometric functions repeat every 360°, so adding or subtracting a multiple of 360° to a solution gives another solution. For example, in Example 2(a) it was found that the solutions of the equation $\cos\theta = 0.8$ in the interval 0° to 360° are approximately 37° and 323°. So some other approximate solutions are, for example,

The tangent function repeats every 180°, but that means that it also repeats every 360°.

$$37° + 360° = 397°,$$
$$37° - 2 \times 360° = -683°,$$
$$323° - 360° = -37°.$$

2.3 Finding the angle of inclination of a line

In this subsection you'll look at another topic which illustrates that putting together ideas covered in different parts of the module can be helpful, and also involves the related angles and trigonometric equations that you learned about in the last subsection.

As you saw in Unit 6, one way to specify the slope of a straight line is to give its gradient.

Another way is to use the idea of angle, which was a topic in Unit 8. The **angle of inclination** of a line is its angle measured anticlockwise from the positive direction of the x-axis, when the line is drawn on a pair of axes *with equal scales*. In Figure 17, the angle of inclination of a straight line is marked as θ. An angle of inclination is always between 0° and 180°.

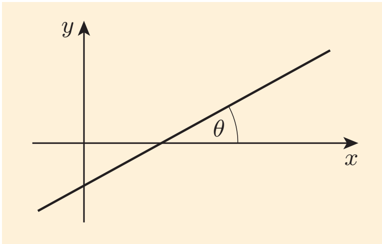


Figure 17 The angle of inclination of a straight line

The gradient of a line ought to be related in some way to its angle of inclination, but how?

To answer this question, first consider a line with a *positive* gradient – it has an *acute* angle of inclination θ, as shown in Figure 18(a).

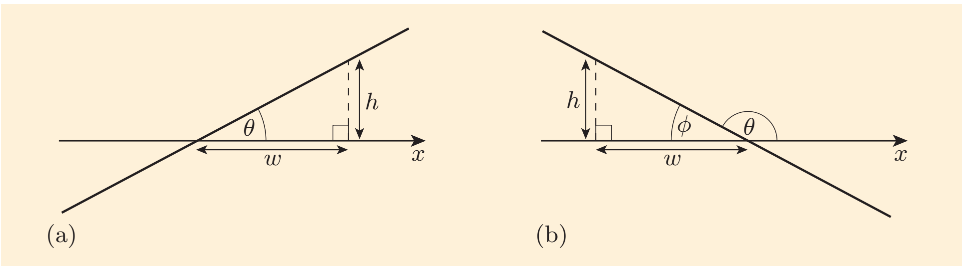


Figure 18 (a) A line with a positive gradient. (b) A line with a negative gradient.

A dashed line perpendicular to the x-axis has been added to the diagram in Figure 18(a), making a right-angled triangle with width w and height h. The gradient of the slant line is given by

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{h}{w},$$

but you can also see that

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{h}{w}.$$

So

$$\text{gradient} = \tan\theta.$$

This simple equation uses trigonometry to relate the ideas of gradient and angle of inclination.

In fact, the same equation also holds for lines with *negative* gradients. Such a line has an *obtuse* angle of inclination θ, as shown in Figure 18(b). In this diagram you can see that

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{-h}{w} = -\frac{h}{w}.$$

The tangent of θ can be worked out from the tangent of the related acute angle ϕ. The diagram shows that

$$\tan\phi = \frac{\text{opp}}{\text{adj}} = \frac{h}{w}.$$

Also, since $\theta = 180° - \phi$, it follows from the related angles diagram that the two angles θ and ϕ have the same tangent values, except possibly for the signs. The angles ϕ and θ are acute and obtuse angles, respectively, so they are first- and second-quadrant angles, respectively, and hence it follows from the CAST diagram that their tangents are positive and negative, respectively. So

$$\tan\theta = -\tan\phi = -\frac{h}{w}.$$

So, again, gradient $= \tan\theta$. This useful relationship is summarised below.

Gradient and angle of inclination of a straight line

For any straight line with angle of inclination θ,

$$\text{gradient} = \tan\theta.$$

Remember that the angle of inclination is measured when the line is drawn on axes *with equal scales.*

Example 3 *Finding the equation of a line with a given angle of inclination*

Find the equation of the straight line shown below, which passes through the point $(0, -1)$ and whose angle of inclination is 30°.

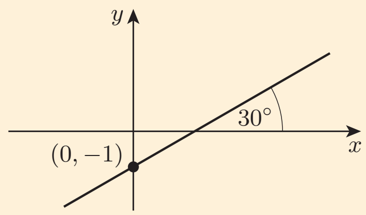


Table 2 Sines, cosines and tangents of special angles

θ	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	–

The special angles table is also given in the Handbook.

Solution

The equation of the line is $y = mx + c$, where m is the gradient and c is the y-intercept.

The y-intercept is -1, since the line passes through the point $(0, -1)$.

To find the gradient, use the fact that gradient $= \tan\theta$, where θ is the angle of inclination. You can find the tangent of 30° either by using your calculator or from the special angles table (Table 2).

The gradient is $\tan 30° = \dfrac{1}{\sqrt{3}}$, so the equation of the line is

$$y = \frac{1}{\sqrt{3}}x - 1.$$

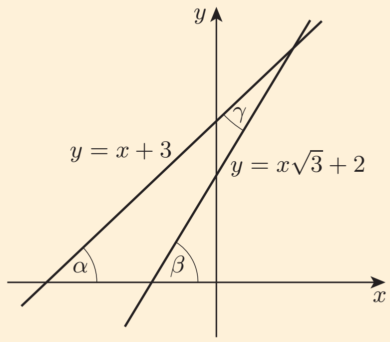
Now have a go at the following activity – it uses the same ideas but the other way round!

Activity 13 *Working with angles of inclination*

The diagram below shows the lines

$$y = x + 3 \quad \text{and} \quad y = x\sqrt{3} + 2,$$

drawn on axes with equal scales. The angles of inclination of these lines are α and β, respectively, and the acute angle between the lines is γ, as shown.



(a) Write down the gradient of each line.

(b) Use your answers to part (a) to calculate the angles α and β in degrees. You can use either your calculator or the table of special trigonometric values (Table 2).

(c) Hence work out the angle γ between the lines.

The second equation here can also be written as

$$y = \sqrt{3}x + 2.$$

However, to avoid the possibility of this being misread as

$$y = \sqrt{3x} + 2,$$

it is preferable to write it as shown.

In the next activity you are asked to find the angle of inclination of a line whose gradient is negative. Since the gradient is negative, the angle of inclination is obtuse, and it is the solution θ of an equation of the form

$$\tan\theta = \text{a negative number}.$$

You can use the methods for solving trigonometric equations that you met in the last subsection to find this obtuse angle.

Activity 14 *Finding an angle of inclination when the gradient is negative*

Consider the straight line with equation

$$y = -2x + 1.$$

(a) Sketch the line on a graph, using equal scales on the axes.

(b) Suppose that the line has angle of inclination θ. Write down a trigonometric equation involving θ, and solve it to find the value of θ to the nearest degree. (Remember that an angle of inclination is between 0° and 180°.)

2.4 Solving trigonometric equations in radians

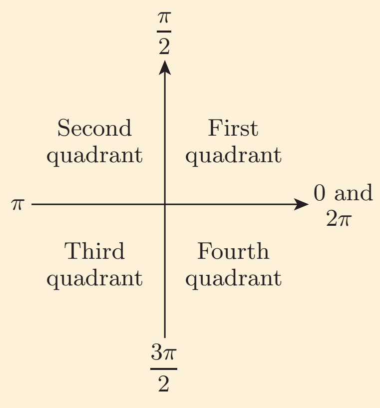


Figure 19 The quadrants

If you go on to higher-level mathematics modules, then you will often work with angles in radians rather than degrees. So in this subsection you will have a chance to practise solving trigonometric equations in radians. You can use exactly the same method as in Subsection 2.2, just with the angles converted to radians.

Remember that 2π radians is the same as 360°, so the boundary values of the quadrants in radians are

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2} \quad \text{and} \quad 2\pi,$$

as shown in Figure 19.

Figure 20(a) repeats the CAST diagram from earlier, and Figure 20(b) shows the related angles diagram in radians.

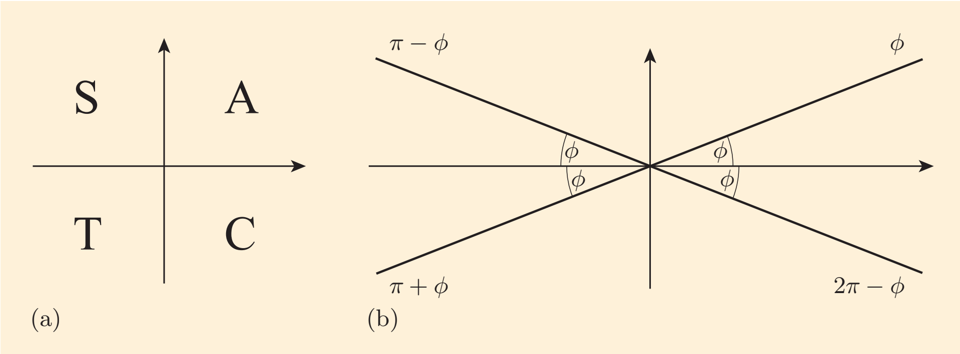


Figure 20 (a) The CAST diagram. (b) The related angles diagram in radians.

In the example below a trigonometric equation is solved in radians.

Example 4 *Solving a trigonometric equation in radians*

Find all the solutions between 0 and 2π of the equation

$$\sin\theta = -\frac{\sqrt{3}}{2},$$

giving exact answers in radians.

Solution

The equation is

$$\sin\theta = -\frac{\sqrt{3}}{2}.$$

Use the CAST diagram to find the quadrants of the solutions.

The sine of θ is negative, so θ is a third- or fourth-quadrant angle.

Find the related first-quadrant angle, either by using your calculator or by recognising that $\sqrt{3}/2$ appears as a sine value in the special angles table (Table 2 on page 202).

The related first-quadrant angle is

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

Use the related angles diagram to find the solutions.

The solutions are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3},$$

$$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

(Check: A calculator gives

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \quad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}.)$$

Here is a similar activity for you to try. When you do it, don't solve the equations in degrees and then convert the solutions to radians. Instead, work with radians throughout, as in Example 4; this will be useful practice for doing mathematics at higher levels. Make sure that your calculator is set to use radians.

Activity 15 *Solving trigonometric equations in radians*

Find all the solutions between 0 and 2π radians of the following equations, giving your answers in radians. In parts (a) and (b) give exact answers, and in part (c) give answers to three significant figures.

(a) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (b) $\tan\theta = -\sqrt{3}$ (c) $\cos\theta = 0.4$

Finally in this subsection, notice that you can use the CAST diagram and the related angles diagram, together with the table of special angles (Table 2 on page 202), to find the exact values of the sines, cosines and tangents of some more angles.

For example, the special angles table tells you that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so, by the related angles diagram, the cosine of each of the following angles is either $\frac{1}{2}$ or $-\frac{1}{2}$:

$$\pi - \frac{\pi}{3}, \quad \pi + \frac{\pi}{3}, \quad 2\pi - \frac{\pi}{3}.$$

These angles simplify to

$$\frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad \frac{5\pi}{3}.$$

They lie in the second, third and fourth quadrants, respectively, so, by the CAST diagram,

$$\cos\left(\frac{2\pi}{3}\right) = -\tfrac{1}{2}, \quad \cos\left(\frac{4\pi}{3}\right) = -\tfrac{1}{2}, \quad \cos\left(\frac{5\pi}{3}\right) = \tfrac{1}{2}.$$

You can use this method to find the exact value of the sine, cosine or tangent of any angle that is linked by the related angles diagram to one of the special angles.

2.5 Solving trigonometric equations by using sketch graphs

When you have to solve a trigonometric equation, an alternative to using the CAST diagram and the related angles diagram is to work with sketch graphs of the sine, cosine and tangent functions.

This method is not the same as the graphical method for solving equations that you used earlier in this unit – it does not involve using graphs to read off approximate solutions. Instead, it involves using sketch graphs to tell you the same information about related angles that you can obtain from the CAST diagram and the related angles diagram. So you do not need accurate graphs, but only sketches showing the basic shapes.

You might find the graphs of the sine, cosine and tangent functions easier to remember and use than the CAST diagram and the related angles diagram. If you cannot remember the shapes of these graphs, then you can look them up in the Handbook. You should work with graphs that have 0°, 90°, 180°, 270° and 360° marked on the θ-axis (or the equivalents in radians), as these are the boundary values between the four quadrants.

The method is demonstrated in the example below, for angles in degrees. Remember that it is just an alternative to the method that you saw in Subsection 2.2 – if you find that you prefer the earlier method, then stick with that!

Tutorial clip

Example 5 *Using sketch graphs to solve trigonometric equations*

Find all the solutions between 0° and 360° of the following equations.

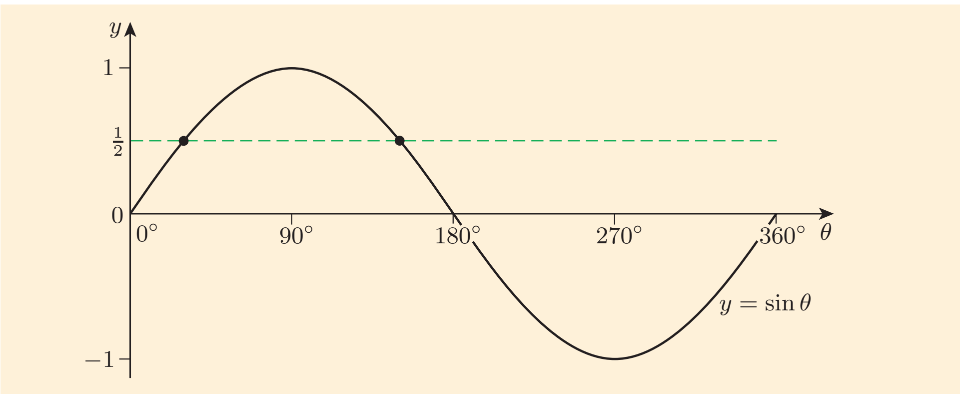
(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\theta = -\frac{1}{2}$

Solution

(a) The equation is

$$\sin\theta = \tfrac{1}{2}.$$

Sketch the graph of $y = \sin\theta$ in the interval 0° to 360°. Draw the horizontal line at $y = \frac{1}{2}$, and mark the crossing points.

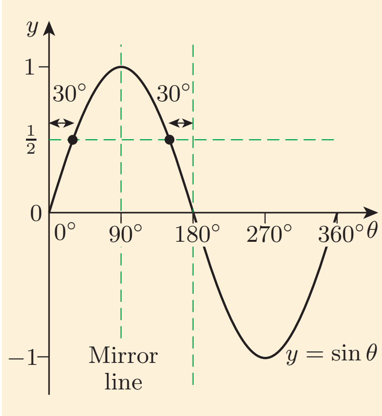


Find one solution, either by using your calculator or from the special angles table (Table 2 on page 202).

One solution is

$$\theta = \sin^{-1}\left(\tfrac{1}{2}\right) = 30°.$$

This solution is the θ-coordinate of the left-hand dot on the sketch graph. To find the other solution, which is the θ-coordinate of the right-hand dot, use the fact that the graph of $y = \sin\theta$ has mirror symmetry in the vertical line at $\theta = 90°$, as shown in the margin.



The second solution is as far below 180° as the first solution is above 0°, so the second solution is

$$180° - 30° = 150°.$$

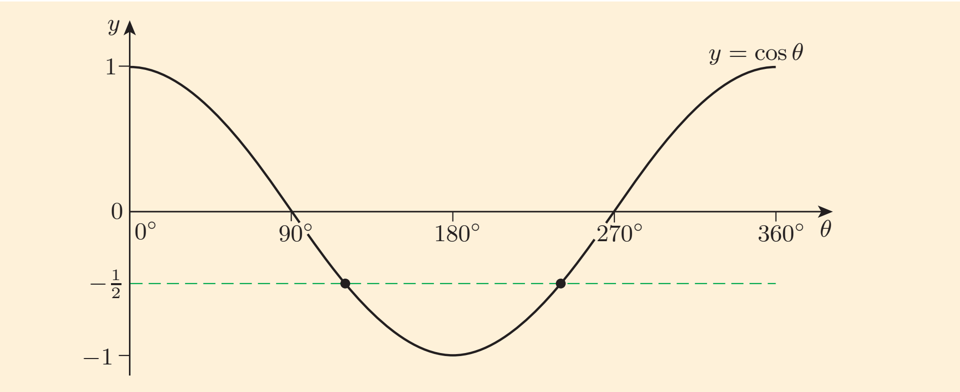
(Check: A calculator gives

$$\sin 30° = \tfrac{1}{2}, \quad \sin 150° = \tfrac{1}{2}.)$$

(b) The equation is

$$\cos\theta = -\tfrac{1}{2}.$$

Sketch the graph of $y = \cos\theta$ in the interval 0° to 360°. Draw the horizontal line at $y = -\frac{1}{2}$, and mark the crossing points.



Use your calculator to find one solution.

One solution is

$$\theta = \cos^{-1}\left(-\tfrac{1}{2}\right) = 120°.$$

This solution corresponds to the left-hand dot on the sketch graph. To find the other solution, use the fact that the graph of $y = \cos\theta$ has mirror symmetry in the vertical line at $\theta = 180°$.

The second solution is as far above 180° as the first solution is below 180°.

The first solution is $180° - 60°$, so the other solution is

$$180° + 60° = 240°.$$

(Check: A calculator gives

$$\cos 120° = -\tfrac{1}{2}, \quad \cos 240° = -\tfrac{1}{2}.)$$

Here is an activity in which you can practise using sketch graphs to help you to solve trigonometric equations. Remember that you need to draw only rough sketch graphs, as their purpose is just to give you information about the symmetry of where the solutions occur. Remember also to set your calculator to use degrees.

Activity 16 *Using sketch graphs to solve trigonometric equations*

Use sketch graphs to find all the solutions between $0°$ and $360°$ of the following equations. Give your answers to part (a) to the nearest degree, and exact answers to part (b).

(a) $\cos\theta = 0.7$ (b) $\tan\theta = 1$

As you have seen, when you use a sketch graph to solve a trigonometric equation of the type considered in this section, the first step after sketching the graph is to use your calculator (or the table of special angles) to find one solution of the equation. If the number on the right-hand side of the equation is negative, then the solution given by your calculator may not be in the range $0°$ to $360°$. In this situation, you can find a solution in the range $0°$ to $360°$ by adding $360°$.

You might like to practise using sketch graphs to solve trigonometric equations in radians rather than degrees. You could try to solve the equations in Activity 15 on page 205 by this method.

In this section, you have reviewed and extended your skills in solving equations, you have learned how to solve simple trigonometric equations, and you have also seen the importance of linking together different mathematical ideas in order to tackle more complicated problems.

3 Abstract mathematics

Mathematics is a fascinating subject in its own right, and it has been appreciated for its beauty for centuries and throughout many different cultures. An important part of mathematics is asking questions, for example about properties of numbers or geometric figures, investigating ideas that arise from these questions, then proposing conjectures, and finally proving that the conjectures are true. Although initially these abstract ideas often do not appear to have any practical use and are studied for their own interest, important applications are sometimes developed later. This section gives a flavour of some of these aspects of mathematics, as well as illustrating that mathematics is everywhere. The first subsection considers some mathematical puzzles from Japan; the second subsection is about some numerical results that can be shown to be true generally; and the final subsection takes a geometric look at paper-folding and its practical applications.

3.1 Japanese puzzles

In Japan, during the Edo period (1603–1868), mathematical ideas flourished in a magnificent way! Instead of mathematics being studied just by experts and in schools, it became available to many people through thousands of *sangaku puzzles*. These were beautiful wooden tablets that were hung as offerings in Shinto shrines and Buddhist temples throughout the country. An example is shown in Figure 21. Each sangaku contained theorems or problems with their answers, but often not the full explanations or solutions. So they became challenges for anyone who could

Sangaku is pronounced as 'san-gak'. The word means 'mathematical tablet'. The oldest surviving sangaku dates from 1683.

understand the language used and wanted to try them. Although many of the tablets were produced by the samurai (the military nobility), anyone could make one, and women and children produced sangaku problems and solutions too. Many of the sangaku described complicated geometric problems.

By this time, most of the samurai were no longer warriors, but government officials and courtiers.

Figure 21 A sangaku tablet featuring eleven geometric problems

The next activity is about a sangaku that was hung in the Kurasako Kannon temple in the Iwate Prefecture in the north of the main island, Honshu, in 1743.

Activity 17 *Solving a sangaku puzzle*

The question on the sangaku is as follows.

> There are 50 chickens and rabbits. The total number of feet is 122. How many chickens and how many rabbits are there?

(Source: Fukagawa, H. and Rothman, T. (2008) *Sacred mathematics*, Princeton, Princeton University Press.)

Note that here '50 chickens and rabbits' means 50 animals altogether; some are rabbits and some are chickens.

(a) Let the number of chickens be c and the number of rabbits be r. Write down an expression for the total number of chickens and rabbits, and an expression for the total number of feet, both in terms of c and r. Hence write down two simultaneous equations in c and r, and solve them to find c and r.

(b) Find another way of solving the puzzle that involves arithmetic, but not algebra.

Being able to solve a sangaku was quite an achievement, but a much better way to impress your fellow mathematicians, and to thank the gods for your mathematical prowess, was to create one. That meant thinking up your own mathematical question and then solving it – just like research mathematicians do today, but on a smaller scale.

Researchers in pure mathematics usually try to use mathematical reasoning to answer a whole family of questions at once, rather than a particular question.

Constructing your own questions can be an excellent way of deepening your understanding of a topic, because you have to think carefully about what restrictions might apply as well as providing a full solution.

One way to start thinking about new questions is to see whether a problem that you have already met can be extended, or ask yourself what would happen if you changed one aspect of the problem in some way. The next activity challenges you to do this, by using Activity 17 as a starting point.

Activity 18 *Creating your own puzzle*

The question in Activity 17 was based on two kinds of animal, one with two feet and the other with four feet.

(a) Consider the following question.

> There are 6 cats and dogs. The total number of feet is 24. How many cats are there?

Explain why there is no single answer to this question.

(b) Think of a scenario for a new question. For example, you could choose 5 insects (each with 6 legs) and 4 spiders (each with 8 legs).

Calculate how many animals and how many legs (or feet) there are for your example.

Then write down a question. For example: 'There are 9 insects and spiders. The total number of legs is 62. How many insects and how many spiders are there?'

(c) Write down two equations that describe your scenario, and solve them. Check that your answers are correct.

(d) Now try varying the problem. For example, instead of saying how many animals there are, you could say how many eyes there are, or you could vary the body parts from feet and legs, or you could vary the number of kinds of animal. Write down a new question and see if you can solve it.

If you like, post your question to the module forum and see if anyone can solve it!

Some of the most beautiful sangaku puzzles posed complicated geometric questions involving circles, triangles and squares. Often Pythagoras' Theorem was needed to solve them.

If you ever needed convincing that problems can be solved in different ways, then Pythagoras' Theorem is an excellent example, as there are hundreds of different proofs from all over the world. In Unit 8, you saw two of these; one proof (in the text) used similar triangles, and another (in the video) involved rearranging four triangles and calculating areas.

In the next activity, you are asked to construct another proof of Pythagoras' Theorem. This proof uses the same geometric construction as in the Unit 8 video, but then uses some of the algebra that you met in Unit 9 to prove the result, rather than a geometric argument.

Activity 19 *Proving Pythagoras' Theorem*

In this activity you are asked to prove Pythagoras' Theorem for a general right-angled triangle with shorter sides of lengths a and b, and hypotenuse of length c, as shown in Figure 22.

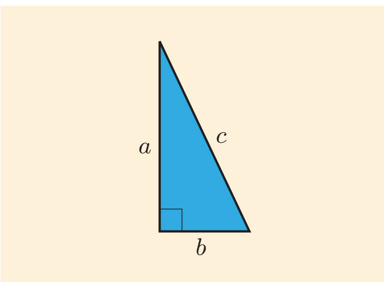
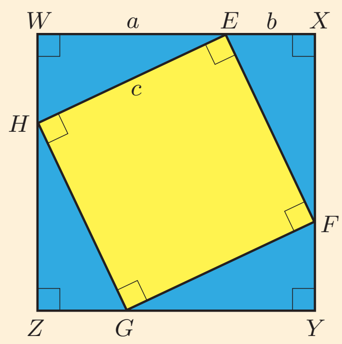


Figure 22 A right-angled triangle

In the following diagram, four identical copies of this triangle are arranged to make a square, $WXYZ$, with sides of length $a + b$. By the symmetry of the situation, the four angles formed by the hypotenuses of the triangles are all equal, so the hypotenuses form a smaller square, $EFGH$, with sides of length c.



(a) Write down an expression for the area of the large square, $WXYZ$, in terms of a and b.

(b) Write down an expression for the area of triangle WEH, in terms of a and b.

(c) Deduce an expression in terms of a and b for the area of the smaller square, $EFGH$, and simplify it.

(d) Using your answer to part (c), deduce Pythagoras' Theorem.

In Activity 19, you saw that applying some of the algebra that you met in Unit 9 to the geometric construction given in Unit 8 led to a new way to prove Pythagoras' Theorem. Making connections between different mathematical ideas like this is a powerful way to solve problems, as you also saw in the last section.

3.2 Proving results

In Unit 1, you found by looking at some examples that if you add up the first n odd numbers, then the sum always seems to be n^2, and you saw how a geometric argument involving arrays of dots could be developed to prove that this result is true in general. This investigation illustrated some steps that are useful for investigating any number pattern or puzzle. These steps are:

1. Look at some numerical examples first.
2. Spot a pattern.
3. Make a conjecture.
4. Prove your conjecture.

You saw a similar approach with the think-of-a-number puzzles in Unit 5, and with some of the number patterns in Unit 9. Most of the conjectures were proved by using algebra.

The steps above are used in the next example, and again the conjecture is proved by using algebra.

Example 6 *Squaring an odd number*

Investigate what happens when an odd number is squared. Make a conjecture and then prove that your conjecture is true for any odd number.

Solution

Try some numerical examples first.

$1^2 = 1$; $3^2 = 9$; $(-5)^2 = 25$; $9^2 = 81$.

Spot a pattern.

All these answers are odd.

Make a conjecture.

A conjecture is: 'The square of an odd number is always odd.'

Prove your conjecture.

Consider any odd number. Since it is not divisible by 2, it is equal to $2n + 1$, for some integer n.

The square of the odd number is $(2n + 1)^2 = 4n^2 + 4n + 1$.

Now $4n^2$ and $4n$ are both even numbers, because 4 is divisible by 2 and n is an integer. Hence $4n^2 + 4n$ is an even number. So the square of the odd number is one more than an even number, so it is an odd number.

This proves the conjecture that the square of an odd number is always odd.

Can you suggest a similar conjecture for the square of an even number?

The algebraic proof in the example above shows that the conjecture holds for *any* odd number, not just for the numerical examples that have been calculated.

Try a similar approach in the following activity.

Activity 20 *Conjecturing and proving*

Consider the following instructions.

- Take any two numbers that sum to 1.
- Square the larger and add the smaller.
- Square the smaller and add the larger.
- Compare the two results – which is larger?

(a) Try some numerical examples first, then make a conjecture.

(b) Let the larger number be x. Write down an expression for the smaller number in terms of x. Prove your conjecture using algebra.

This subsection has highlighted how algebra can be used to prove general results.

3.3 Geometry everywhere

You have already seen several applications of geometry, for example in architecture, surveying, design and navigation. However, there are many more applications of geometry and plenty of discoveries still to be made, and this subsection gives a glimpse of just a few of these.

In the first part of this subsection, you are invited to apply some of the geometric ideas that you have met already to origami – the Japanese art of paper-folding.

Folding squares into triangles

The next two activities pose a challenge: using only the geometric properties of squares and triangles, can you prove that the angles in a triangle made by folding a square in a particular way are certain sizes?

Each of the activities involves a square of paper, which you can make from a rectangular piece, such as A4, in the way shown in Figure 23.

The method for folding a square into a triangle in Activity 21 was designed by Kunihiko Kasahara, a Japanese origami expert who is interested in the relationship between mathematics and origami.

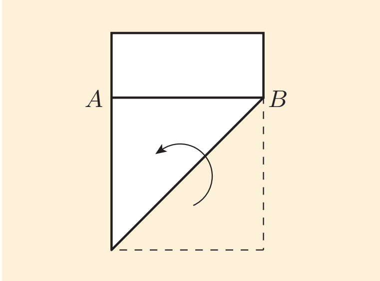
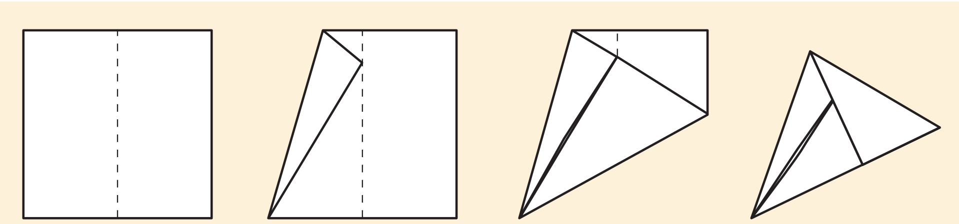


Figure 23 To make a square from a rectangle, fold the rectangle as shown and cut along the line AB.

Activity 21 *Folding a square into a triangle*

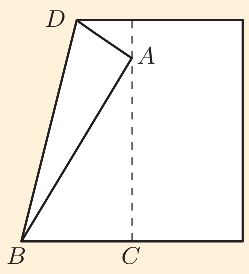
Take a square of paper and fold it carefully as shown in the diagrams below. In step 3, you should find that two of the folded-over edges lie in a straight line – this is because you have placed two right angles together. In step 4, you should find that when you fold along this straight line, the corner of the square that was at the top right folds down to lie exactly on the edge of the paper shape. You will see why this happens shortly!

The method used in Activity 21 is given in Franco, B. (1999) *Unfolding mathematics with unit origami*, Emeryville, Key Curriculum Press.



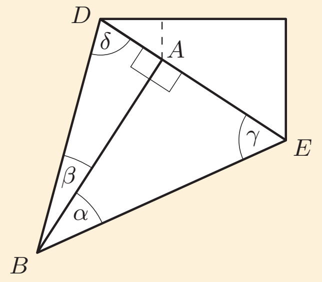
(a) The diagram below shows the folded square after step 2. Suppose that the sides of the square have length 2 units. Write down the lengths of AB and BC. By using a trigonometric ratio, or otherwise, find $\angle ABC$.

The sides of the square have been chosen to have length 2 units, rather than 1 unit, to simplify the arithmetic. You saw a similar approach when the sine, cosine and tangent of special angles were found in Unit 12.



(b) The diagram overleaf shows the folded square after step 3. Use your answer to part (a) to find the angles α, β, γ and δ.

Hint: First find the angle α. To find β, notice that there are two layers of paper between the edges DB and EB, so, since an angle of a square is 90°, it follows that $2(\alpha + \beta) = 90°$.



(c) Use the size of one of the angles that you found in part (b) to explain why, when you make the fold in step 4, the corner of the square that was at the top right lies exactly on the edge of the paper shape.

(d) Write down the sizes of the three angles of the final triangle.

You can use the triangle constructed in Activity 21 to measure angles of 45°, 60° and 75°, or any combination of these angles. In fact, you can use it to check your progress in the next activity.

Activity 22 *Folding a square into an equilateral triangle*

(a) What size are the angles in an equilateral triangle?

(b) Take a square of paper. Can you fold it into an equilateral triangle, without using a ruler or protractor? You might find it helpful to look back at Activity 21 to see how you constructed angles of different sizes.

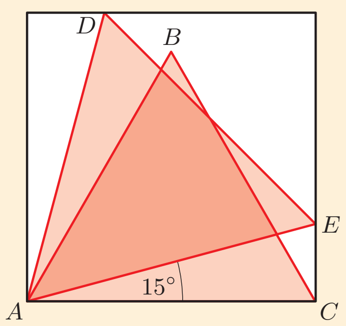


Figure 24 An equilateral triangle larger than the one in the solution to Activity 22

Another interesting question is: Can you fold a square into a larger equilateral triangle than that given in the solution to Activity 22?

You can indeed do this – $\triangle ADE$ in Figure 24 is obtained by rotating $\triangle ABC$ anticlockwise about the bottom left-hand corner of the square and extending the lengths of its three sides, so it is larger than $\triangle ABC$. You might like to think about how you could fold the square into this larger triangle without using a protractor – it is possible!

Applications of geometry

From Activities 21 and 22, you can see that there is a lot of mathematics even in simple paper-folding. Research into the mathematics of origami has shown that there are four mathematical rules that apply to the fold pattern for any origami design. This has resulted in many complicated designs being developed.

Some origami designs have practical applications, especially in situations where it is important to fold items compactly before they are used. For example, in 1995 a large folding solar panel was packed compactly into the shape of a parallelogram and then expanded in space, on a Japanese satellite known as the Space Flyer Unit. The design was based on ideas by Japanese professor Koryo Miura at Tokyo University, who has also designed effective ways of folding maps.

More recently, in 2007, folding techniques were used by engineers at the University of California to develop a very thin optical device to replace the conventional lens in a digital camera, which could then be used in a mobile

phone, for example. Another application of origami is in the design of car airbags, so that they can fit within a small space, yet be deployed quickly and effectively when needed.

You may have got the impression from the last few paragraphs that the only people who make new discoveries in geometry, or indeed in mathematics generally, are university academics! However, this isn't the case. In 1975, American housewife Marjorie Rice discovered four previously unknown tiling patterns using pentagons, which are shown in Figure 25. Although she had studied mathematics only at school, she had been inspired to try to find new tiling patterns after reading an article in the magazine *Scientific American*.

The article referred to here is Gardner, M. (1975) 'On tessellating the plane with convex polygon tiles', *Scientific American*, vol. 233, no. 1, pp. 112–17.

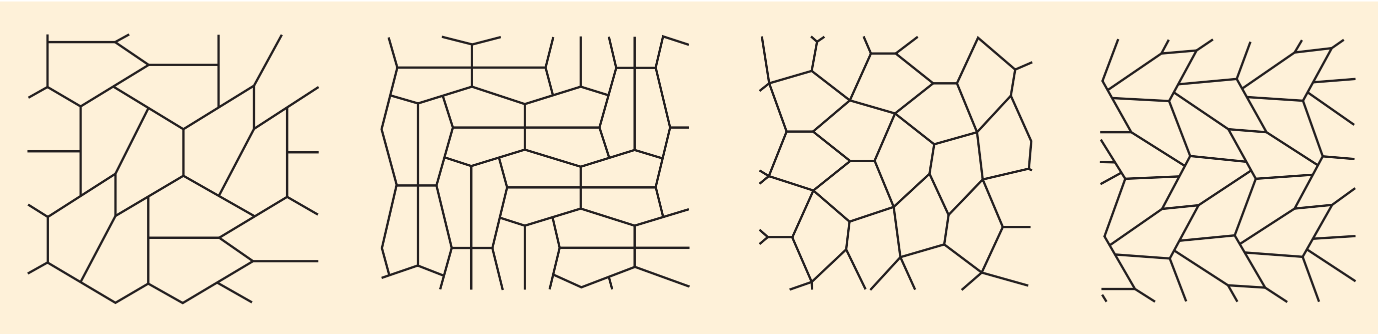


Figure 25 Four new pentagonal tilings discovered by Marjorie Rice

Occasionally new advances are made because someone has come up with a better attempt at solving a long-standing problem. In 1887, Lord Kelvin considered the problem of how to partition space into cells of equal volume in such a way that the area of surface between the cells is as small as possible. This problem has since become known as the *Kelvin problem*. Lord Kelvin suggested that the solution to the problem might be a structure based on a 14-sided polyhedron known as a *truncated octahedron*, as shown in Figure 26. No better structure was found for over a century, until in 1993 two Irish physicists, Denis Weaire and Robert Phelan, discovered a new structure with a smaller area of surface between the cells than in Lord Kelvin's structure. Weaire and Phelan's structure uses two different types of polyhedra, one with 14 sides and the other with 12 sides. It is now thought that Weaire and Phelan's structure is the best possible – in other words, that it is the solution to the Kelvin problem – but at the time of writing, this has not been proved.

There is some information about Lord Kelvin, William Thompson, in Unit 7, Subsection 1.1.

A **polyhedron** is a solid with flat faces.

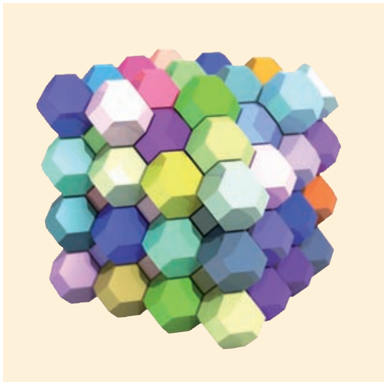


Figure 26 Lord Kelvin's structure

The architects who designed the Water Cube (Figure 27), the building for the swimming pool used in the 2008 Beijing Olympics, were inspired by Weaire and Phelan's ideas. Despite its name, the building is actually a cuboid, not a cube!

Figure 27 The Water Cube in Beijing, built for the 2008 Olympics

Figure 28 A gömböc (pronounced 'goemboets', which is something like 'gumboots')

Another recent geometric discovery is the gömböc. This is a very particular type of solid object that was constructed in 2006 by the Hungarian scientists Gábor Domokos and Péter Várkonyi. The solid is made from a uniformly dense material, but it has the amazing property that whichever way you try to make it balance, it always self-rights itself to a standard position. There are different shapes of gömböc – one example is shown in Figure 28. Some turtle shells have similar properties to gömböcs, which may help the turtles to get back onto their feet when they are turned over!

Geometric discoveries are not just about structures and shapes – whole new geometries have been discovered too.

Some new geometries

In Unit 8 you learned that all of the results that are used in geometry, such as the fact that opposite angles of a parallelogram are equal, can be proved by starting with a small number of initial assumptions about elements of geometry, such as points and lines. The initial assumptions are usually called *axioms* or *postulates*. A suitable list of axioms is set out in Euclid's *Elements*, and the geometry that is developed from them is called *Euclidean geometry*. This is the familiar geometry that you have learned in MU123, and that is generally used in everyday applications.

Euclidean geometry can be two-dimensional – the geometry that you do in a plane – or it can be extended to three dimensions – the geometry that you do in three-dimensional space. (It can even be extended to more dimensions, but the resulting geometries do not apply to 'real-life' space.)

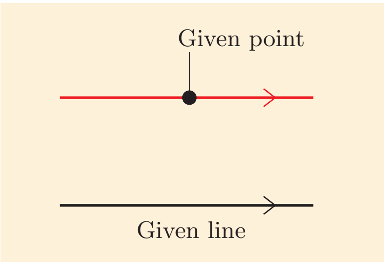


Figure 29 Euclid's parallel postulate

One of Euclid's axioms for two-dimensional geometry is known as the *parallel postulate*, and it is equivalent to the following statement:

> Given a line and a point that does not lie on the line, there is exactly one line through the given point that is parallel to the given line.

Figure 29 shows that this assumption appears to be true – given any line and point on a plane, where the point does not lie on the line, you can draw exactly one line through the given point that does not cross the given line.

In the nineteenth century, mathematicians discovered that it was possible to replace the parallel postulate with either of two different assumptions, with each of these two possibilities leading to a whole new type of geometry that made sense in itself but was very different from Euclidean geometry. As with Euclidean geometry, these new types of geometry can be two-dimensional or three-dimensional (or can be extended to more dimensions).

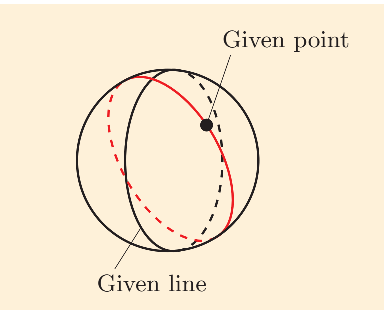


Figure 30 Elliptic geometry

In the two-dimensional version of one of these types of geometry, called *elliptic geometry*, the new assumption is that, given a line and a point that does not lie on the line, there are *no* lines through the given point that are parallel to the given line – in other words, *every* line through the given point crosses the given line. This type of geometry can be visualised by thinking of points as the points on a sphere, and lines as the circles of maximum diameter on the sphere – these are called *great circles*. (Another way to think of the great circles is that if you cut down through a great circle you cut the sphere in half.) The great circles are the 'paths of shortest distance' on the sphere, just as straight lines are the paths of shortest distance on a plane. Now if you draw a 'line' (great circle) on the sphere and choose a point not on this 'line', then any 'line' that you draw through the point will cross the original 'line', as illustrated in Figure 30. So there are no 'lines' that pass through the given point that are parallel to the given 'line'.

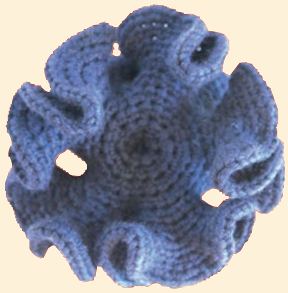
In the two-dimensional version of the other type of geometry, called *hyperbolic geometry*, the new assumption is that there are at least *two* lines that pass through any given point and are parallel to a given line. One way to visualise this type of geometry is by thinking of a type of surface called a *hyperbolic plane*, in a similar way to the sphere for elliptic geometry. You think of points as the points on the hyperbolic plane, and lines as the paths of shortest distance on the hyperbolic plane. Unfortunately a hyperbolic plane is itself less easy to visualise than a sphere, but one surprising way to gain some insights into this type of surface, and hence into hyperbolic geometry, is by crocheting part of such a surface!

Figure 31 Parallel lines on a hyperbolic plane

Crochet is a handicraft similar to knitting that involves looping and intertwining yarn with a hooked needle. Figure 31 illustrates a crochet model for part of a hyperbolic plane, with four 'lines' sewn on it. Three of the 'lines' pass through the same point, and they are all parallel to the fourth 'line': they do not cross it because of the way that the surface curves.

Some natural objects exhibit some of the properties of hyperbolic planes, and of other types of 'hyperbolic surfaces'. For example, Figure 32 shows the similarities between the edges of a marine flatworm and part of a hyperbolic surface.

(a)



(b)

Figure 32 A marine flatworm and part of a hyperbolic surface

Some crocheted hyperbolic surfaces look very similar to certain types of coral. In 2005, the science writer Margaret Wertheim started an arts project in the USA known as the 'Hyperbolic Crochet Coral Reef' that uses crocheted hyperbolic surfaces. She decided to crochet a giant coral reef to draw attention to the damage done to coral reefs by global warming. This stunning exhibition has now toured all over the world, introducing many people to both hyperbolic geometry and crochet, as well as highlighting the plight of coral reefs. Part of the crochet coral reef is shown in Figure 33, and on the module website there are some links to other websites where you can find out more about it.

A book entitled *Crocheting Adventures with Hyperbolic Planes*, by Dr Daina Taimina, won the Diagram prize for the world's oddest book title in 2010. The prize has been awarded annually by *The Bookseller* magazine since 1978.

Figure 33 Part of the crochet coral reef

So whether you are looking at paper-folding, airbags, crochet or coral reefs, mathematics really is everywhere!

4 Using mathematics practically

An important aspect of mathematics is using it to describe aspects of the world, and solve associated real-life problems. For small problems, you may be able to use an appropriate mathematical technique straightaway. However, as you have seen in the module, in many cases assumptions have to be made to simplify a situation before any mathematics can be applied. Then an appropriate mathematical model is used to describe the situation and obtain results. The results have to be interpreted and checked carefully to ensure that the mathematical model is appropriate. Sometimes, further modifications to the model are then required. This process is summarised in the modelling cycle shown in Figure 34.

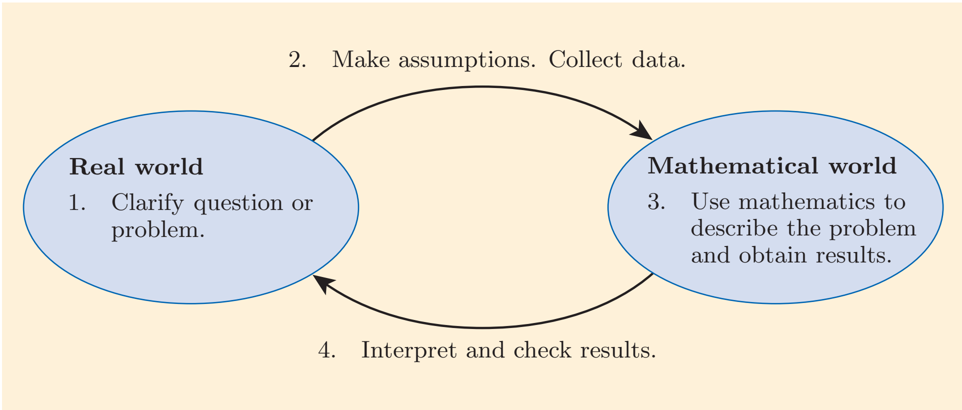


Figure 34 The mathematical modelling cycle

In Unit 2, you saw how this process was used in developing some route-planning software. You have also seen how a similar cycle can be used when carrying out a statistical investigation such as the ESP experiment in Unit 11.

In this section, you'll learn how trigonometric functions can be used in mathematical modelling. These functions are useful for modelling situations that are cyclical, such as circular motion, the heights of the tides or the amount of daylight on different days of the year.

The first subsection considers how the height of a gondola on a ferris wheel changes as the wheel rotates. A gondola on a ferris wheel is a viewing capsule suspended on an axle, which hangs down as the wheel rotates. You will see how a formula for the height of a gondola can be developed, using the trigonometry that you learned in Unit 12 and earlier in this unit. Since the assumptions needed for modelling this situation, and the interpretation of the mathematical results, are relatively straightforward, this subsection concentrates on the mathematical ideas rather than the modelling cycle. The formula developed belongs to a family of trigonometric functions that are related to the sine and cosine functions, and you will explore this family of functions in the next subsection, using Graphplotter. These functions are useful in several modelling situations, and in the final subsection you'll learn how you can apply them and the modelling cycle to predict the height of the tide in certain situations.

Angles are measured in radians rather than degrees throughout this section, as this is usual when trigonometric functions are used for modelling. Remember that a full turn (360°) is the same as 2π radians, so a half-turn (180°) is π radians, a quarter-turn (90°) is $\pi/2$ radians, and so on.

4.1 Modelling the motion of a ferris wheel

Large ferris wheels are popular attractions, as they give open views of the surrounding area as well as being fun to ride on. For example, Figure 35 shows the Big Wheel that was constructed in Centenary Square in Birmingham, UK, in November 2009. As the wheel slowly rotates, the gondolas travel up and then down, and extensive views of the city are visible from gondolas above a certain height.

Figure 35 The Big Wheel, Birmingham, UK

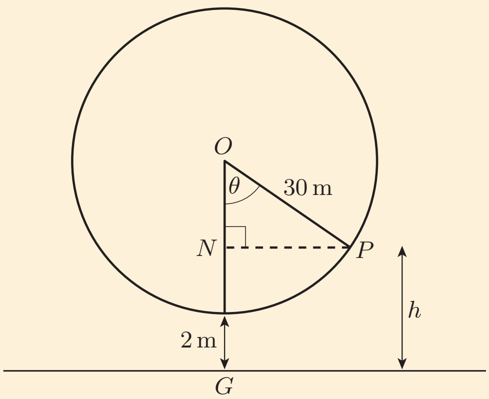
In this subsection we'll consider the example of a ferris wheel that has a diameter of 60 metres, with the lowest point of the rim 2 metres above the ground. Once all the occupants of the gondolas have embarked, the wheel rotates continuously at a constant speed, taking 20 minutes to make a complete revolution. The best views are visible from gondolas whose points of attachment to the rim of the wheel are 20 metres or more above the ground.

Now suppose that you want to know the times when the best views are visible during the 20-minute journey of a gondola from its lowest point and back again.

By looking at the geometry of the wheel, you can work out a formula for the height of the gondola (which you can take to mean the height of its point of attachment) at any time during its circular journey.

Let's begin by working out a formula for the height of the gondola in terms of the angle θ through which it has rotated from its lowest point. Once we've found this formula, we'll use it to find a formula for the height of the gondola in terms of the time since it was at its lowest point.

Figure 36 shows the situation when the gondola has rotated through an angle θ from its lowest point. The point P is the position of the point of attachment of the gondola, and h is its height above the ground in metres.



This diagram is not drawn to scale.

Figure 36 The height h of the point P after a rotation through an angle θ

The diagram shows the situation when the angle θ is acute, so h is equal to $OG - ON$.

The distance OG is the radius of the wheel plus the height of its lowest point above the ground. The radius of the wheel is 30 metres, since its diameter is 60 metres, and the lowest point is 2 metres above the ground. Hence

$$OG = 30 + 2 = 32.$$

The distance ON can be found by using a trigonometric ratio in the right-angled triangle OPN, as follows:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{ON}{30},$$

so

$$ON = 30\cos\theta.$$

Hence, when θ is acute,

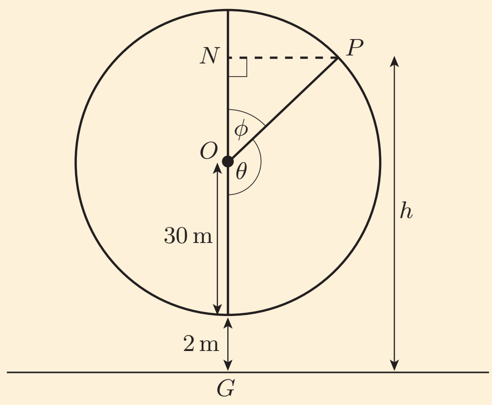
$$h = OG - ON = 32 - 30\cos\theta.$$

In fact, because of the way that the cosine of an angle is defined, this formula holds for any angle θ, whether it is acute or of any other size. In the next activity you are asked to check that it holds for obtuse angles.

Activity 23 *Showing that the formula for h holds when θ is obtuse*

The diagram below shows the situation when the angle θ is obtuse. The acute angle between OP and the upward vertical from O is labelled as ϕ.

This diagram is not drawn to scale.



(a) Use this diagram to work out a formula for h in terms of ϕ.

(b) Notice that $\theta = \pi - \phi$. Use the facts about related angles that you learned in Subsection 2.2 to show that $\cos\theta = -\cos\phi$.

(Hint: Use an argument similar to the one that shows that $\tan\theta = -\tan\phi$ on page 202.)

(c) Hence show that the formula found in part (a) is equivalent to the formula $h = 32 - 30\cos\theta$ that was found for acute angles θ.

An argument similar to that in Activity 23, based on related angles, holds for an angle θ of any size.

So we now have a formula for h in terms of θ. However, what we really want is a formula for h in terms of the time t minutes since P was the lowest point on the wheel. Such a formula can be found by working out how θ is related to t.

The wheel rotates through 2π radians every 20 minutes. So in 1 minute it rotates through

$$\frac{2\pi}{20} = \frac{\pi}{10} = 0.314 \text{ radians (to 3 s.f.)},$$

and hence in t minutes it rotates through $0.314t$ radians.

So, t minutes after the point P was the lowest point on the wheel, it has rotated through an angle of $\theta = 0.314t$ radians, and therefore its height h metres is given by

$$h = 32 - 30\cos(0.314t).$$

This is the formula for h in terms of t that we wanted.

Now that you have this formula, you can find the times for which the best views are visible. These views are visible when the height is greater than or equal to 20 metres, so one way to find the times is to draw the graph of h against t and read off the values of t for which $h \geq 20$. In the next activity you are asked to use Graphplotter to do this.

Activity 24 *Finding the times during which the best views are visible*

Graphplotter

Use Graphplotter, with the 'Two graphs' tab selected. On the Options page, ensure that 'Grid', 'Axes', 'Trace' and 'Radians' are all ticked. Choose the equations $y = a\cos(b(x - c)) + d$ and $y = mx + c$ from the two drop-down lists.

(a) Enter the appropriate values for the constants a, b, c and d, and then m and c, so that the graphs of the equations $y = 32 - 30\cos(0.314x)$ and $y = 20$ are plotted. Set the minimum and maximum values on the x-axis to 0 and 20, respectively, by entering these numbers into the x min and x max boxes at the bottom right of Graphplotter. Set the minimum and maximum values on the y-axis to suitable numbers.

(b) Hence find the times, after the gondola is at its lowest point, when the best views become visible and when they cease to be visible. Give your answers to the nearest minute.

An alternative way to find the times for which the best views are visible is to use the methods for solving trigonometric equations from Section 2 to find the values of t for which $h = 20$. That is, you have to find the solutions of the equation

$$32 - 30\cos(0.314t) = 20.$$

To do this, you can first rearrange the equation as follows:

$$12 - 30\cos(0.314t) = 0$$
$$30\cos(0.314t) = 12$$
$$\cos(0.314t) = 0.4.$$

Then you can use the methods for solving trigonometric equations to find the values of θ (in radians) between 0 and 2π for which $\cos\theta = 0.4$. In fact, you were asked to do this in Activity 15(c) on page 205 – the values are

$$\theta = 1.1592\ldots \quad \text{and} \quad \theta = 2\pi - 1.1592\ldots = 5.1239\ldots.$$

You need to find the values of θ between 0 and 2π because θ is the angle of rotation and you are interested in the times when the best views are visible during one rotation of the wheel. These times will correspond to angles of rotation between 0 and 2π.

This means that the required values of t are given by

$$0.314t = 1.1592\ldots \quad \text{and} \quad 0.314t = 5.1239\ldots.$$

These equations give

$$t = \frac{1.1592\ldots}{0.314} = 3.6919\ldots = 4 \text{ (to the nearest whole number)}$$

and

$$t = \frac{5.1239\ldots}{0.314} = 16.3181\ldots = 16 \text{ (to the nearest whole number).}$$

So, as you should have found in the solution to Activity 24, the best views become visible approximately 4 minutes after the gondola was at its lowest point, and cease to be visible approximately 16 minutes after the gondola was at its lowest point.

In Activity 24 you were asked to plot the height of the gondola for one complete revolution of the ferris wheel. If you plot the height for a longer period of time, then you obtain a graph like the one in Figure 37.

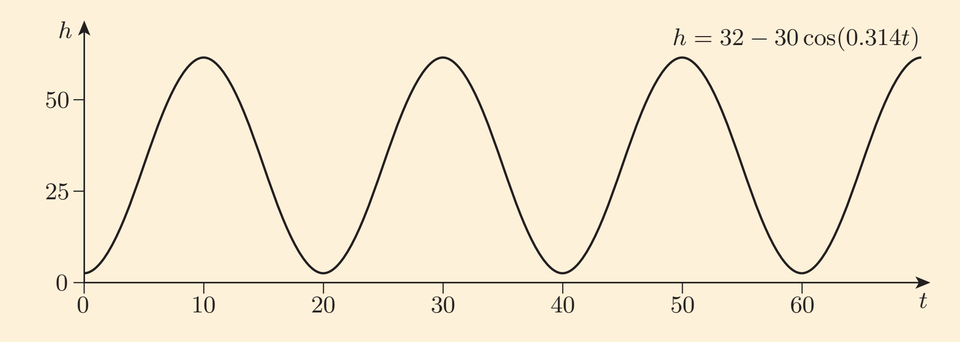


Figure 37 The height h metres of the gondola plotted against the time t minutes after it is at its lowest point

The graph in Figure 37 appears to have a shape similar to that of a sine or cosine curve, but it lies at a different position on the axes and it is stretched horizontally and vertically. A curve that can be obtained from the graph of the sine function by shifting, stretching or compressing it horizontally or vertically is called a **sinusoidal curve**, and a function whose graph is a sinusoidal curve is called a **sinusoidal function**.

4.2 Exploring sinusoidal functions

In this subsection you will explore the graphs of equations of the form

$$y = a\sin(b(x - c)) + d \quad \text{and} \quad y = a\cos(b(x - c)) + d,$$

where a, b, c and d are constants. Functions with rules of these forms where a and b are non-zero are called **general sine functions** and **general cosine functions**, respectively, and you will see that they are always sinusoidal functions. In the next activity, you'll see how the values of the constants a, b, c and d affect the shapes of the graphs of these equations.

The y-value of any sinusoidal function *oscillates* between a minimum value and a maximum value, and the shape of the graph is continually repeated. The length on the x-axis that it takes for the graph to repeat is called the **period** of the graph, as you saw in Unit 12. Any section of the graph covering this length on the x-axis is referred to as an **oscillation** or *cycle*, as illustrated in Figure 38.

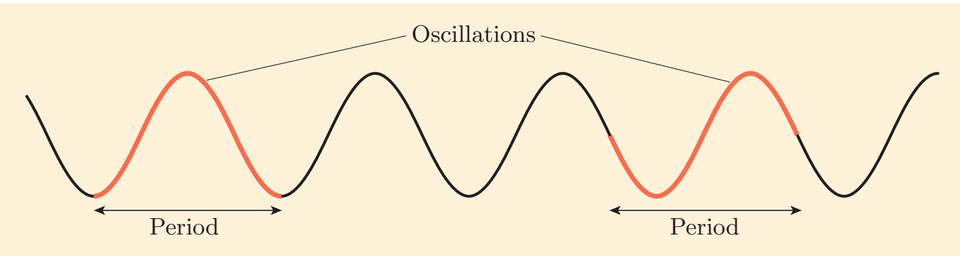


Figure 38 Oscillations of a sinusoidal function

Activity 25 *Exploring the graphs of general sine and cosine functions*

Graphplotter

Use Graphplotter, with the 'One graph' tab selected. On the Options page ensure that 'Grid', 'Axes' and 'Radians' are all ticked, and the other options are not ticked.

Choose the function $y = a\sin(b(x - c)) + d$ from the drop-down list. The default values of the constants are $a = 1$, $b = 1$, $c = 0$ and $d = 0$, so you should see the graph of $y = \sin x$.

Choosing the function also has the effect of setting the x-axis to the interval from -2π to 2π, which is about -6.3 to 6.3. Set x min $= 0$, so that the x-axis is set to the interval from 0 to 2π instead; then you will see one complete oscillation.

(a) Set y min and y max to -5 and 5, respectively. Use the slider to change the value of a, and note the effect on the graph. What do you notice about the value of a and the minimum and maximum values of y?

What happens if you change the value of a to its negative (for example, from 2 to -2)?

(b) Set $a = 1$, so you see the graph of $y = \sin x$ again. Then change the value of b to 2, 3 and 4 in turn. For each of these values, count how many complete oscillations there are in the interval from 0 to 2π.

Can you make a conjecture about the value of b and the number of oscillations in this interval, if b is a positive integer? Check your conjecture by trying another value of b. Does your conjecture hold for fractional values of b, such as 2.5?

What happens if b is negative? In particular, what happens if you change the value of b to its negative?

(c) Set $b = 1$, so you see the graph of $y = \sin x$ again. Also set x min $= -6.3$, so that the x-axis is set to the interval -2π to 2π again (approximately).

Use the slider to increase and decrease the value of c. Notice that the graph appears to move to the right if c is increased, and to the left if c is decreased.

To investigate the relationship between the graphs of $y = \sin x$ and $y = \sin(x - c)$, first tick the 'Trace' option and set $c = 0$. Then concentrate on the point on the curve that is at the origin when $c = 0$, and find the coordinates of the position that this point moves to when you choose a new value of c. Start by choosing small values for c, such as 0.1, 0.2, 0.3 and their negatives, typing the values into the box rather than using the slider.

What do you think is the relationship between the graphs of $y = \sin x$ and $y = \sin(x - c)$?

(d) Set $c = 0$, so you see the graph of $y = \sin x$ again. What do you think would be the effect on the graph of changing the value of d? Untick 'Trace' on the Options page and tick 'y-intercept' instead. Then use the slider to vary d, to see if your prediction is correct.

(e) Now select the equation $y = a\cos(b(x - c)) + d$ from the drop-down list. Vary the constants a, b, c and d in the same way as you did for the general sine function. How does the graph change?

When you have completed your investigation, read the comments on this activity at the end of the unit.

In Activity 25, you looked at the graphs of general sine and cosine functions, which have equations of the form

$$y = a\sin(b(x-c)) + d \quad \text{and} \quad y = a\cos(b(x-c)) + d. \tag{7}$$

The effects that you saw when you varied the values of a, b, c and d might have been familiar to you from your investigations of other functions, such as quadratic functions. For example, if you replace the variable x in the equation of any function by the expression $x - c$, where c is a constant, then the graph of the function moves right by distance c (the graph moves left if c is negative).

Similarly, if you replace x by bx, where b is a constant, then the graph is compressed or stretched in the x-direction. For a general sine or cosine function, this changes the number of oscillations within a given interval. You saw in the activity that if the constant b in either of equations (7) is positive, then the number of oscillations in the interval from 0 to 2π seems to be b, and if b is negative, then the number of oscillations seems to be the magnitude of b.

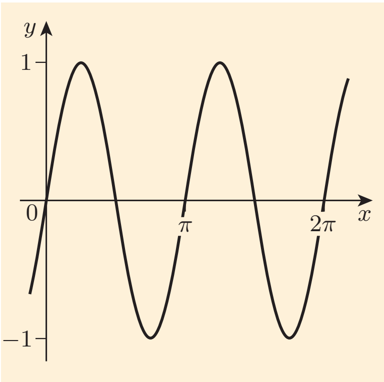


Figure 39 The graph of the equation $y = \sin(2x)$

This finding can be stated concisely by using notation for the magnitude of a number. In general, the magnitude of any number x is denoted by $|x|$. So, for example, $|-5| = 5$ and $|2| = 2$. So the activity seemed to show that the number of oscillations in the interval from 0 to 2π is $|b|$.

Another way to think of this finding is in terms of the *period* of the graph, that is, the length along the x-axis that the graph takes to repeat. The period of a general sine or cosine graph can be worked out by dividing the length of a given interval on the horizontal axis by the number of oscillations in that interval. For example, in Figure 39 there are two oscillations in the interval from 0 to 2π, so the period of this graph is $2\pi/2 = \pi$. So another way of stating the finding about b is that the period of the graph seems to be $2\pi/|b|$. This finding is indeed true in general, as stated below.

The period of a general sine or cosine function

The period of $y = a\sin(b(x-c)) + d$ or $y = a\cos(b(x-c)) + d$ is $\dfrac{2\pi}{|b|}$.

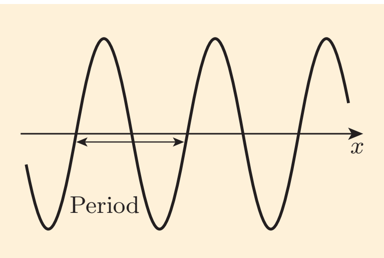


Figure 40 The graph of an equation of the form $y = \sin(bx)$

To see why this result holds, first notice that the period of $y = a\sin(b(x-c)) + d$ or $y = a\cos(b(x-c)) + d$ is the same as the period of $y = \sin(bx)$, since either of the former graphs is obtained from the latter by shifting it horizontally and vertically, and/or stretching or compressing it vertically. (The cosine curve can be obtained from the sine curve by shifting it along the x-axis.)

You can find the period of the graph of $y = \sin(bx)$ by considering where it crosses the x-axis. As illustrated in Figure 40, the period is twice the distance between any two consecutive x-intercepts. The x-intercepts are the solutions of the equation $\sin(bx) = 0$. Now $\sin\theta = 0$ whenever θ is a multiple of π (that is, 180°), so $\sin(bx) = 0$ whenever bx is a multiple of π. So one pair of consecutive x-intercepts is given by

$$bx = 0 \quad \text{and} \quad bx = \pi,$$

that is, $x = 0$ and $x = \pi/b$. The second of these x-intercepts could be either larger or smaller than the first, depending on whether b is positive or negative, but in either case the distance between the two x-intercepts is $\pi/|b|$, so the period is $2\pi/|b|$, as stated in the pink box.

To illustrate this result, consider the equation $y = \sin(3(x+1))$. The value of the constant b is 3, so the graph of this equation has period $2\pi/3$.

Activity 26 *Finding the periods of general sine and cosine functions*

Find the periods of the graphs of the following equations.

(a) $y = \sin 4x$ (b) $y = 3\sin 5x - 2$ (c) $y = \cos \dfrac{x}{2} - 1$

(d) $y = 2\cos(-x)$

Before going on, let's check that the formula for the period works for the ferris wheel example that you met earlier. The formula that was worked out for the height of a gondola was $h = 32 - 30\cos(0.314t)$. This equation is of the form $y = a\sin(b(x-c)) + d$, with $b = 0.314$ (and x and y replaced by t and h, respectively), so the period of the graph is

$$\frac{2\pi}{0.314} = 20.01 \text{ (to 2 d.p.)}.$$

The variable t is measured in minutes, so this means that the values for the height repeat every 20.01 minutes, approximately. This is as expected, because the model was based on the assumption that the time for a complete revolution of the wheel is 20 minutes. The slight discrepancy arises from the rounding that was done when the formula was worked out.

You have seen that general sine functions and general cosine functions give exactly the same family of graphs, namely the sinusoidal curves. Because of this, the remainder of this subsection will concentrate on general sine functions, that is, functions with rules of the form

$$y = a\sin(b(x-c)) + d. \tag{8}$$

Also, exactly the same family of graphs is obtained if the constants a and b in equation (8) are allowed to take only *positive* values, instead of either positive or negative values. This is because the effect obtained by changing the value of a or b to its negative can also be obtained by just shifting the graph horizontally, by changing the value of c. So, from now on, this subsection will concentrate on equations of form (8) where a and b are *positive*.

Each of the constants a, b, c and d in equation (8) relates to a key feature of the graph of the equation, as illustrated in Figure 41 below and explained in the box overleaf.

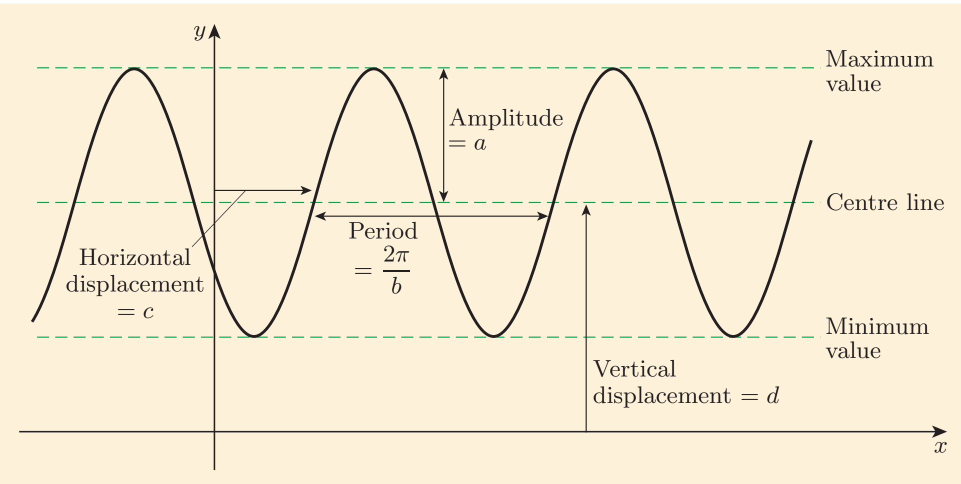


Figure 41 The graph of a general sine function

The graph of a general sine function

The graph of the equation

$$y = a\sin(b(x-c)) + d,$$

where a and b are positive, and c and d can take any values, has the following features.

- a is the **amplitude**: the distance between the centre line and the maximum (or minimum) value.
- b determines the *period*, which is equal to $2\pi/b$.
- c is the **horizontal displacement**: the amount that the graph of $y = a\sin(bx) + d$ is shifted to the right to obtain the graph of $y = a\sin(b(x-c)) + d$. (The shift is to the left if c is negative.)
- d is the **vertical displacement**: the amount that the centre line is shifted up from the x-axis. (The shift is down if d is negative.)

Also,

- the minimum value is $d - a$ (the vertical displacement minus the amplitude)
- the maximum value is $d + a$ (the vertical displacement plus the amplitude).

The following example illustrates how you can sketch the graph of a general sine function.

Example 7 *Sketching the graph of a general sine function*

Sketch the graph of the equation

$$y = 4\sin(2x-1) + 1.5.$$

Solution

Make sure that the equation is in the form of the general equation.

Factorising part of the given equation gives

$$y = 4\sin(2(x-0.5)) + 1.5.$$

Find the values of a, b, c and d, and use them to find the features of the graph.

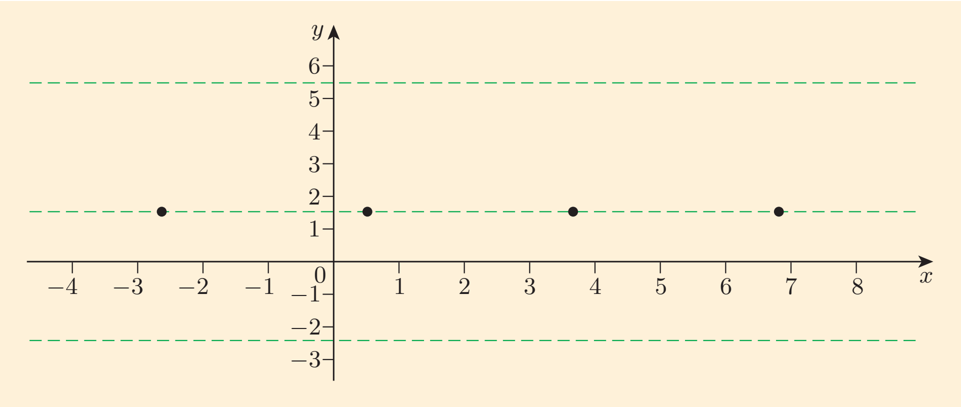
Here $a = 4$, $b = 2$, $c = 0.5$ and $d = 1.5$. So

$$\begin{aligned}
&\text{amplitude} = 4,\\
&\text{period} = 2\pi/2 = \pi \approx 3.14,\\
&\text{horizontal displacement} = 0.5,\\
&\text{vertical displacement} = 1.5,\\
&\text{minimum value} = 1.5 - 4 = -2.5,\\
&\text{maximum value} = 1.5 + 4 = 5.5.
\end{aligned}$$

Choose suitable axis limits. The x-axis should include a few period lengths, so choose, say, -4 to 8. The y-axis should include the minimum and maximum values, so choose, say, -3 to 6.

Mark the centre line, minimum and maximum as dashed lines.

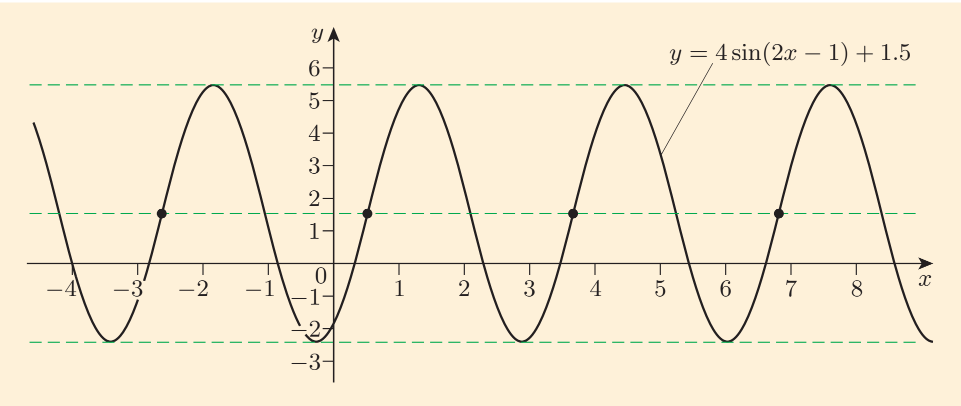
Mark the point on the centre line that is the distance of the horizontal displacement, which is 0.5, to the right of the y-axis (it would be to the left if the horizontal displacement were negative). Mark other points on the centre line at intervals of a period length, which is approximately 3.14.



The points marked on the centre line have the following x-coordinates:

$0.5 - 3.14 = -2.64$,
0.5,
$0.5 + 3.14 = 3.64$,
$0.5 + 2 \times 3.14 = 6.78$.

Draw an oscillation between each pair of marked points. The graph goes up and then down to the right of each marked point (since a and b are positive).



Here is a similar activity for you to try.

Activity 27 *Sketching the graphs of general sine functions*

Sketch the graphs of the following equations.

(a) $y = \sin 3x$ (b) $y = \sin(x + 2)$ (c) $y = 2\sin(x - 1.5) - 1$

Hint: In part (b), remember that $x + 2$ is the same as $x - (-2)$.

Figure 42 A harbour tide gauge

4.3 Modelling the tides

In the video for Unit 12, you saw how the height of the tide varies both throughout the day and throughout the year. The physical model developed to predict the tide heights didn't take account of factors such as weather conditions (which can affect tide heights), but nevertheless the model proved very useful for producing tide tables. Knowing the height of the tide at different times is important so that people on the coast or at sea can plan their activities safely. For example, when navigating a ship or boat into a harbour, it is important to check that the water is deep enough so that the vessel does not run aground. For small boats, knowing when the tide is high can also make loading and unloading cargo easier. Checking the time for the high tide before you embark on a coastal walk may prevent you becoming stranded too!

Tide tables showing the high and low tides are published in newspapers and are available on the internet, but knowing the height of the tide at other times is often important too. For example, if you plan to walk out to an island that gets cut off by the sea, then you will need to know the times when the tide is low enough for the island to be accessible by foot, not just when the high tide occurs.

So if you know the times of the high and low tides at a particular location on a particular day, is it possible to develop a simple mathematical model to predict the height of the tide at other times during the day?

To illustrate how trigonometry and the modelling cycle can be used to help to answer this type of question, let's consider a particular example, namely:

> What was the height of the tide at different times of the day on 14 December 2009 at Milford Haven in South Wales?

The next step in the modelling cycle, after posing the question, is to collect relevant data and to make some assumptions so that the situation can be described mathematically.

A tide table for Milford Haven gave the information shown in Table 3.

Table 3 High and low tides in Milford Haven on 14 December 2009

Time	Type	Height (metres)
04:21	High	6.2
10:42	Low	1.7
16:45	High	6.2
23:06	Low	1.6

Source: Tide tables on the BBC Weather website at www.bbc.co.uk/weather/coast/tides

Over a day, the height of the tide rises and falls twice. The cyclical nature of these changes suggests that it may be possible to use a general sine function as a model. So a useful assumption to make is that the height of the tide, over the 24 hours of 14 December 2009, can be modelled by a function of the form $h = a\sin(b(t-c)) + d$, where t is the time since midnight in hours and h is the height of the tide in metres.

Now that the data have been collected and the assumptions have been made, the next step in the modelling cycle is to sort out the mathematics; that is, we need to decide on the values for a, b, c and d, and then use this model to make predictions. The values of a, b, c and d should be chosen to fit the data in the tide table, at least approximately, and you can do this by using what you have learned about how the features of the graph of a general sine function correspond to the values of the constants in the equation.

The constant a in the equation of a general sine function is the amplitude, which is half the distance between the minimum and maximum values. So you can choose an appropriate value for a by first choosing appropriate values for the minimum and maximum values. Each of the high tides in Table 3 has a height of 6.2 metres, so the maximum value should be chosen to be 6.2. However, the low tides have different values: 1.7 metres and 1.6 metres. So either of the values 1.7 or 1.6, or their average, could be chosen to be the minimum value. However, since the first low tide occurs close to the middle of the time period for which the model is intended to apply, it is best to choose the minimum value to be the height in metres of this low tide, which is 1.7. This gives the difference between the minimum and maximum values as $6.2 - 1.7 = 4.5$. So the amplitude a should be chosen to be $\frac{1}{2} \times 4.5 = 2.25$.

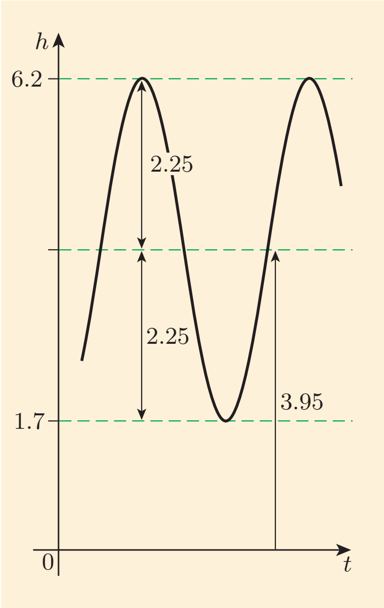


Figure 43 The vertical displacement chosen for the model

Since the amplitude is 2.25 and the minimum value is 1.7, the vertical displacement d must be $1.7 + 2.25 = 3.95$, as shown in Figure 43.

An appropriate value for the constant b can be chosen by first choosing an appropriate value for the period, the length along the horizontal axis that the graph takes to repeat itself. This is the same as the length between any two consecutive minimum values, or between any two consecutive maximum values. In this model the variable on the horizontal axis, t, represents time, so the period will be a time. The data in Table 3 show that the time between the two high tides is the same as the time between the two low tides – it is 12 hours and 24 minutes in each case – so the period should be chosen to be this time. Before this time is used to calculate the value of b, it has to be converted from hours and minutes to hours in decimal form. The time is $\left(12 + \frac{24}{60}\right)$ hours, which is 12.4 hours in decimal form. Hence

$$\frac{2\pi}{b} = 12.4, \quad \text{so} \quad b = \frac{2\pi}{12.4} = 0.507 \text{ (to 3 s.f.).}$$

An appropriate value for the constant c, which is the horizontal displacement, can be chosen by first choosing an appropriate value for the time at which the first maximum value occurs. This should be chosen to be the time of the first high tide, which is 04:21, or 4.35 hours when converted to hours in decimal form. Figure 44 shows that, for any sinusoidal function (with the variable on the horizontal axis representing time), the horizontal displacement can be found by subtracting one quarter of the period from the time when the first maximum value occurs. In this case, the period is 12.4 hours, so one quarter of the period is $12.4/4 = 3.1$ hours. So the horizontal displacement c should be chosen to be

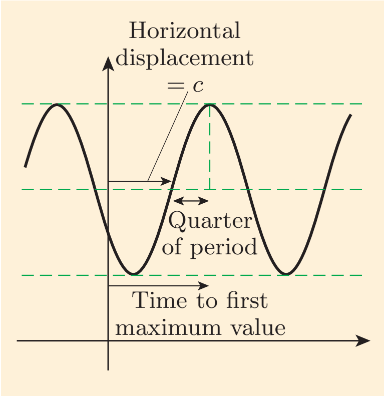


Figure 44 A maximum value occurs a quarter of a period after the horizontal displacement

$$4.35 - 3.1 = 1.25.$$

Now that suitable values of a, b, c, and d have been determined, the equation of the model can be written down as

$$h = 2.25\sin(0.507(t - 1.25)) + 3.95,$$

where t is the time since midnight in hours and h is the height of the tide in metres.

This equation can be used to predict the height of the tide at a particular time, or to predict the times at which the tide is a particular height. In the next activity, you are asked to use the model and then interpret your results. This interpretation is part of the fourth stage of the modelling cycle.

Graphplotter

Activity 28 *Predicting the height of the tide*

Use Graphplotter, ensuring that 'Trace' and 'Radians' are ticked on the Options page. Choose the function $y = a\sin(b(x-c)) + d$ from the drop-down list, and set a, b, c and d to 2.25, 0.507, 1.25 and 3.95, respectively. Set x min $= 0$, x max $= 24$, y min $= 0$ and y max $= 10$.

(a) Enter the value 4.35, which is the time in hours (in decimal form) of one of the high tides, into the x cursor box, and read off the corresponding y-coordinate of the curve. Check that it corresponds to the height of the high tide given in Table 3.

(b) Use the model to predict the times for which the tide is deeper than 4 metres. Give your answers in hours to one decimal place.

The other part of the fourth stage of the modelling cycle is to check the model against reality. Scientists at the United Kingdom Hydrographic Office (UKHO) have developed a computer model that accurately predicts the heights of the tide throughout the day. It takes into account the gravitational effects of the Sun and the Moon, and also uses data that have been collected by tide gauges at various locations on the coast.

The graph in Figure 45 shows the heights of the tide predicted by the UKHO for Milford Haven on 14 December 2009, with the sinusoidal model found in this subsection superimposed in red.

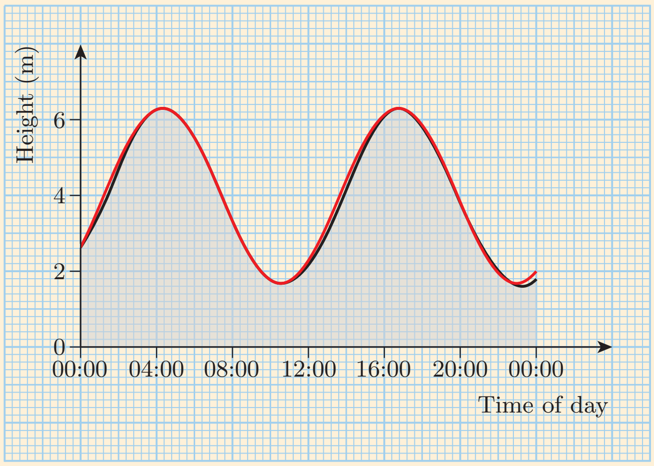


Figure 45 The heights of the tide at Milford Haven on 14 December 2009, with the sinusoidal model found in this subsection superimposed in red

You can see that the sinusoidal model is a close fit to the predictions made by the UKHO, so it is a good model for the height of the tide at Milford Haven on this particular day.

However, the sinusoidal model is not realistic over a longer period of time, such as a year. No single sinusoidal model for the heights of a tide can be realistic over a time period longer than a few days, because the heights of the high and low tides vary, as illustrated in Figure 46. Many other factors need to be taken into account in order to get a realistic long-term model, as you saw in the video for Unit 12.

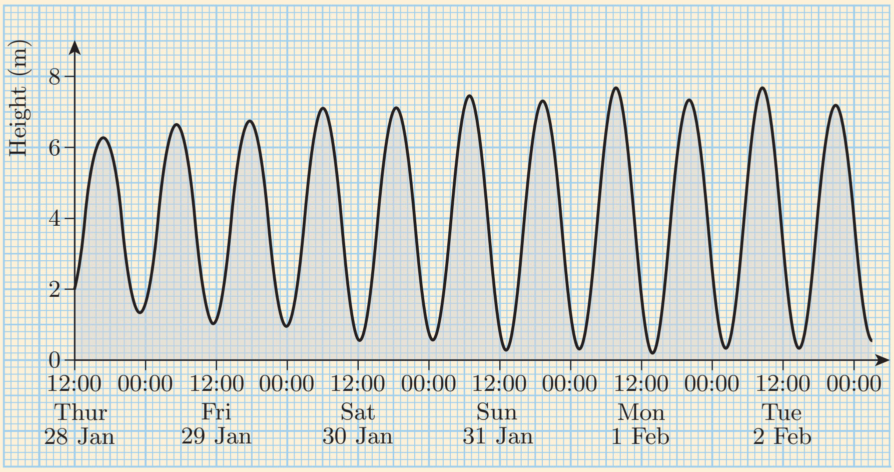


Figure 46 The variation in the height of a tide over several days

In fact, often the heights of the tide cannot be modelled by a sinusoidal function even over a single day. Figure 47 shows the heights of the tide at the entrance of the River Tees in the north-east of England on 11 December 2009. You can see that there is a difference of 0.6 metres between the two low tides, so a model based on a single sinusoidal function will not be accurate in this case.

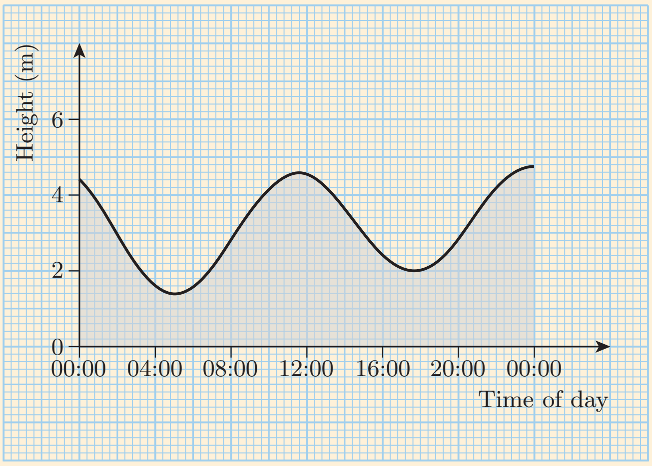


Figure 47 The heights of the tide at the entrance to the River Tees on 11 December 2009

In general, a sinusoidal function will provide a reasonably accurate model for the heights of a tide over a single day, provided that the heights of the two high tides are similar and the heights of the two low tides are also similar.

Your mathematical journey

This final unit of the module has touched briefly on some of the important aspects of your mathematical journey through MU123, highlighting just a few of the many ideas that you have studied. You have also developed some new skills in trigonometry and have seen how some of these can be used in modelling some real-life situations.

With the mathematical techniques that you have learned and the problem-solving skills that you have developed during MU123, you should now be well equipped to use mathematics for many different applications, both in your everyday life and in further studies.

We hope that the module has helped you to enjoy some of the beauty, power and fascination of mathematics, and convinced you to continue on your mathematical journey! If you would like to do this straightaway, then have a look at the links to popular mathematics books and other resources in the Resources section for Unit 14 on the module website. There are links to further modules that you can do in the 'What next?' section in the final week of the study planner.

Learning checklist

After studying this unit, you should be able to:

- understand how the probability of an event happening can be measured
- interpret newspaper articles on risk
- use a variety of ways for solving equations
- find the sine, cosine and tangent of angles measured in degrees and radians
- solve trigonometric equations for angles between $0°$ and $360°$
- use algebra to obtain and prove results
- describe the graph of $y = a\sin(b(x - c)) + d$ for different values of a, b, c and d
- understand the terms amplitude, period, horizontal displacement and vertical displacement
- appreciate the use of trigonometry in models.

Solutions and comments on Activities

Activity 1

The total number of babies is $377 + 367 = 744$. So the percentage of girls is

$$\frac{377}{744} = 0.506\ldots \approx 51\%.$$

This may not be an accurate estimate for the probability of having a girl in the country where the hospital is, as it is unlikely that the sample of births is representative of the whole country. A larger dataset of randomly chosen births throughout the country would give a better estimate.

Activity 2

The probability that the asteroid will collide with the Earth is

$$0.0004\% = \frac{0.0004}{100} = \frac{4}{1\,000\,000},$$

which is 4 in 1 million.

Activity 3

You need to know the probability (as a number) that a person in the general population who does not use a mobile phone will develop mouth cancer. You also need to know how many hours of mobile phone use per day caused the apparent increase in that risk. Other factors to consider are how many days the phone is used on, and perhaps the age and gender of the user.

Activity 4

(a) The risk of getting monkeypox for people who do not eat bananas is 1 in 20. So out of 1000 people who do not eat bananas, the approximate number that you would expect to get monkeypox is

$$1000 \times \frac{1}{20} = 50.$$

(b) The risk of getting monkeypox for people who eat bananas regularly is 1.25 in 20. So out of 1000 people who eat bananas regularly, the approximate number that you would expect to get monkeypox is

$$1000 \times \frac{1.25}{20} = 62.5 \approx 63.$$

(c) The answers to parts (a) and (b) tell you that for a group of 1000 people who eat bananas regularly, you would expect 50 people to get monkeypox anyway, and about 13 more to get monkeypox due to eating bananas. So, on average, eating bananas makes a difference to 13 people out of 1000.

Activity 5

(a) A superficial glance at the pair of graphs might lead you to think that the growth in the number of telephone lines, shown in the left-hand graph, is greater than the growth in the number of mobile phone subscribers, shown in the right-hand graph.

This impression is caused by the different vertical scales on the two graphs.

(b) Scaling up the 1997 picture by a scale factor of 15 to give the 2007 picture creates a misleading impression of the relative sizes of the two numbers of mobile phone subscribers depicted. This is because although the number of subscribers in 2007 is about 15 times the number in 1997, far more than 15 copies of the 1997 picture would fit inside the 2007 picture, giving the impression that the growth in the number of mobile phone subscribers is much greater than it actually is.

(In fact, from what you saw in Unit 8, the number of copies of the 1997 picture that would fit inside the 2007 picture is about $15^2 = 225$.)

Pictograms can be misleading if it is not clear whether the lengths, areas or volumes of the pictures are being compared.

Activity 6

(a) Your reflections are obviously personal, but things to watch out for are:

- clearly explained solutions rather than a string of calculations
- carefully constructed graphs, with clear titles and scales that are easy to read
- correct use of notation and vocabulary
- logically structured arguments
- appropriate rounding.

(b) You might like to write these points directly on the assignment as a reminder for when you start working on the questions, or as a check on any questions that you have already completed.

Activity 7

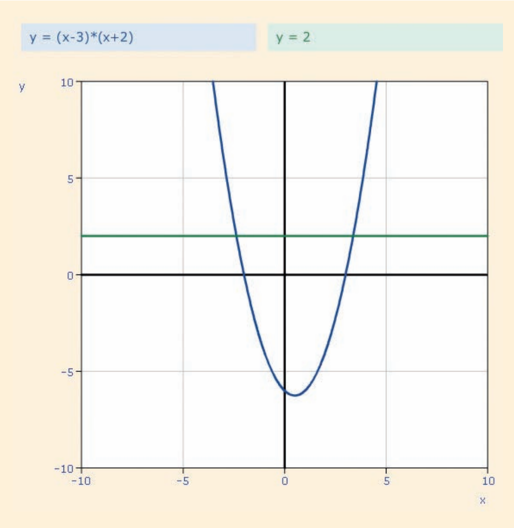
(a) The equation is

$$(x - 3)(x + 2) = 2.$$

You can solve it graphically by plotting the graphs of $y = (x - 3)(x + 2)$ and $y = 2$ on Graphplotter, and finding the x-coordinates of the crossing points.

You can plot the graph of $y = (x-3)(x+2)$ by choosing Custom function, and typing `(x-3)*(x+2)` into the box. (Alternatively you can multiply out the brackets, which gives $y = x^2 - x - 6$, choose the equation $y = ax^2 + bx + c$, and set the values of a, b and c appropriately.)

The resulting graph is shown below.



You can use the procedure described before the activity to find that the two solutions are $x = 3.37$ and $x = -2.37$ (both to 2 d.p.).

(b) The equation is

$$(x-3)(x+2) = 2.$$

It can be solved algebraically as follows.

Expand the brackets:

$$x^2 - 3x + 2x - 6 = 2.$$

Collect the terms all on one side:

$$x^2 - x - 8 = 0.$$

The expression on the LHS does not factorise easily, so we use the quadratic formula, which states that the solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here $a = 1$, $b = -1$ and $c = -8$, so

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-8)}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{33}}{2}. \end{aligned}$$

Hence the solutions are

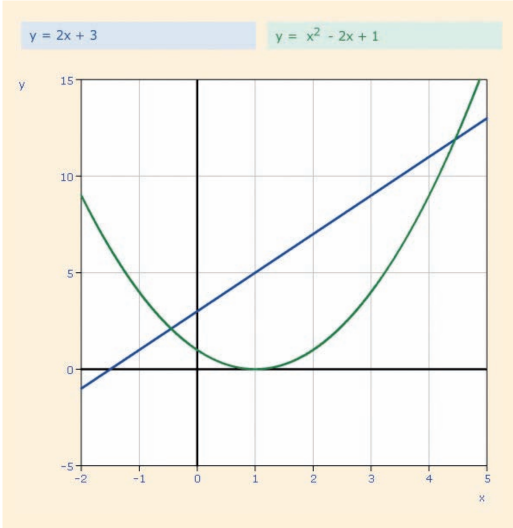
$$x = \frac{1 + \sqrt{33}}{2} \quad \text{and} \quad x = \frac{1 - \sqrt{33}}{2},$$

which are $x = 3.37$ and $x = -2.37$ (both to 2 d.p.).

These are the same values as found in part (a).

Activity 8

(a) The graphs of $y = 2x + 3$ and $y = x^2 - 2x + 1$ are shown below.



To find the coordinates of the crossing points, zoom in on each crossing point in turn, until both the x- and y-coordinates are displayed to at least three decimal places.

For the left-hand crossing point, you can find a point below the crossing point, and a point above the crossing point, that both have coordinates $(-0.45, 2.10)$ when rounded to two decimal places. So these are the coordinates of this crossing point.

Similarly, for the right-hand crossing point, you can find a point below the crossing point, and a point above the crossing point, that both have coordinates $(4.45, 11.90)$ when rounded to two decimal places. So these are the coordinates of this crossing point.

(b) The coordinates of a crossing point are values of x and y that satisfy both equations. That is, they form a solution of both equations simultaneously.

(So the solutions of the simultaneous equations are $x = -0.45,\ y = 2.10$ and $x = 4.45,\ y = 11.90$, to two decimal places.)

Activity 9

The equations are

$$y = x + 1, \tag{9}$$
$$y = x^2 + 2x - 5. \tag{10}$$

The right-hand sides are equal, which gives

$$x + 1 = x^2 + 2x - 5.$$

Simplify:

$$0 = x^2 + x - 6.$$

Factorise:

$$0 = (x+3)(x-2).$$

Hence $x = -3$ or $x = 2$. Substituting these values, in turn, into equation (9) gives $y = -2$ and $y = 3$, respectively. So the solutions are

$$x = -3,\ y = -2 \quad \text{and} \quad x = 2,\ y = 3.$$

(Check: Substituting $x = -3$, $y = -2$ into equation (10) gives

$$\text{RHS} = (-3)^2 + 2 \times (-3) - 5 = 9 - 6 - 5 = -2 = \text{LHS},$$

and substituting $x = 2$, $y = 3$ into the same equation gives

$$\text{RHS} = 2^2 + 2 \times 2 - 5 = 4 + 4 - 5 = 3 = \text{LHS}.)$$

Activity 10

Equations (a) and (b) are in a suitable form for cross-multiplying. (The right-hand side of equation (b) can be thought of as a fraction with denominator 1.)

(a) The equation is

$$\frac{x}{x+1} = \frac{2}{x+3}.$$

Assume that $x \neq -1$ and $x \neq -3$.

Cross-multiply:

$$x(x+3) = 2(x+1).$$

So $x^2 + 3x = 2x + 2$,

and hence $x^2 + x - 2 = 0$.

Factorise:

$$(x+2)(x-1) = 0.$$

So $x + 2 = 0$ or $x - 1 = 0$,

and hence $x = -2$ or $x = 1$.

These answers satisfy the initial assumptions $x \neq -1$ and $x \neq -3$, so the solutions are $x = -2$ and $x = 1$.

(b) The equation is $\dfrac{x}{x+1} = 2$.

Assume that $x \neq -1$.

Cross-multiply:

$$x = 2(x+1)$$

$$x = 2x + 2$$

$$x = -2.$$

This answer satisfies the initial assumption, so the solution is $x = -2$.

(You can cross-multiply in the equation here because the right-hand side can be thought of as the fraction $2/1$. However, because the equation is fairly simple, cross-multiplying isn't really any quicker than just multiplying both sides by the denominator $x + 1$.)

(c) The equation is $\dfrac{x}{x+1} = \dfrac{2}{x} + 3$.

Assume that $x \neq -1$ and $x \neq 0$.

Cross-multiplying cannot be used, so use the usual method of multiplying both sides by an expression that is a multiple of all the denominators.

Multiply by $x(x+1)$:

$$x(x+1)\left(\frac{x}{x+1}\right) = x(x+1)\left(\frac{2}{x} + 3\right)$$

Simplify:

$$x^2 = 2(x+1) + 3x(x+1)$$

$$x^2 = 2x + 2 + 3x^2 + 3x$$

$$0 = 2x^2 + 5x + 2$$

Factorise:

$$0 = (2x+1)(x+2).$$

Hence $2x + 1 = 0$ or $x + 2 = 0$.

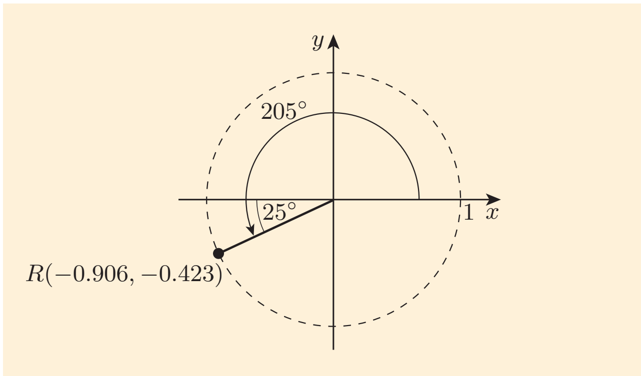
So $x = -\frac{1}{2}$ or $x = -2$.

These answers satisfy the initial assumptions, so the solutions are $x = -\frac{1}{2}$ and $x = -2$.

Activity 11

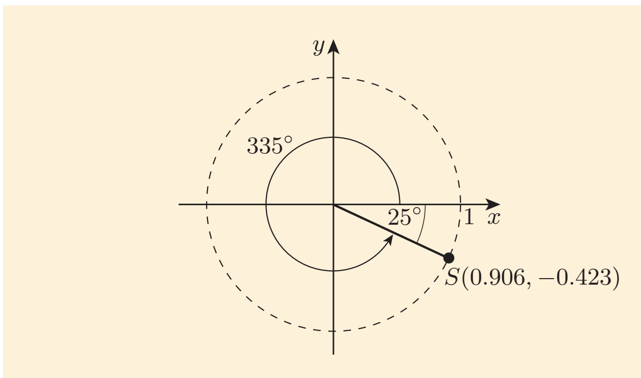
(a) The point R corresponds to the angle

$$180° + 25° = 205°.$$



The point S corresponds to the angle

$$360° - 25° = 335°.$$



(b) The coordinates of R are

$(-0.906, -0.423)$.

The coordinates of S are

$(0.906, -0.423)$.

Both pairs of coordinates are given to three significant figures.

(c) The sine, cosine and tangent of the angle corresponding to R are

$\sin(205°) \approx -0.423, \quad \cos(205°) \approx -0.906$

and

$$\tan(205°) = \frac{-0.422\,61\ldots}{-0.906\,30\ldots} \approx 0.466.$$

Similarly, the sine, cosine and tangent of the angle corresponding to S are

$\sin(335°) \approx -0.423, \quad \cos(335°) \approx 0.906$

and

$$\tan(335°) = \frac{-0.422\,61\ldots}{0.906\,30\ldots} \approx -0.466.$$

All these values are given to three significant figures.

Activity 12

(a) The equation is

$\sin\theta = 0.2.$

The sine of θ is positive, so θ is a first- or second-quadrant angle.

One solution is

$\theta = \sin^{-1}(0.2) = 12°$ (to the nearest degree).

The other solution is

$\theta = 180° - 12° = 168°$ (to the nearest degree).

(Check: A calculator gives

$\sin 12° = 0.207\ldots \approx 0.2,$

$\sin 168° = 0.207\ldots \approx 0.2.$)

(b) The equation is

$\cos\theta = -0.6.$

The cosine of θ is negative, so θ is a second- or third-quadrant angle.

The related first-quadrant angle is

$\theta = \cos^{-1}(0.6) = 53°$ (to the nearest degree).

The solutions are

$\theta = 180° - 53° = 127°$ (to the nearest degree),

$\theta = 180° + 53° = 233°$ (to the nearest degree).

(Check: A calculator gives

$\cos 127° = -0.6018\ldots \approx -0.6,$

$\cos 233° = -0.6018\ldots \approx -0.6.$)

Activity 13

(a) The gradient of the line $y = mx + c$ is m.

So the gradient of the line $y = x + 3$ is 1, and the gradient of the line $y = x\sqrt{3} + 2$ is $\sqrt{3}$.

(b) Hence

$\tan\alpha = 1 \quad \text{and} \quad \tan\beta = \sqrt{3}.$

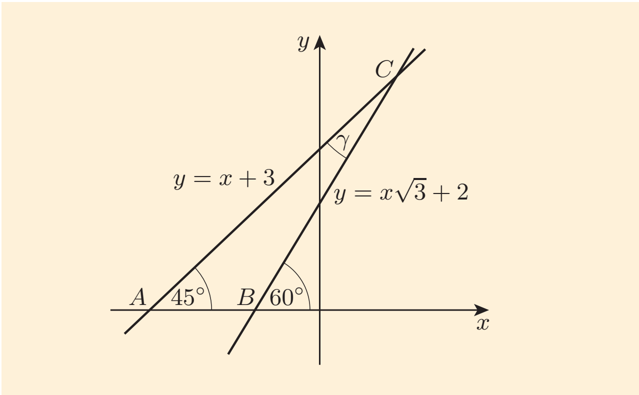
So, since α and β are acute angles,

$\alpha = \tan^{-1}(1) = 45°$

and

$\beta = \tan^{-1}(\sqrt{3}) = 60°.$

(c)



The angles on a straight line add up to 180°. So in the diagram,

$\angle ABC = 180° - 60° = 120°.$

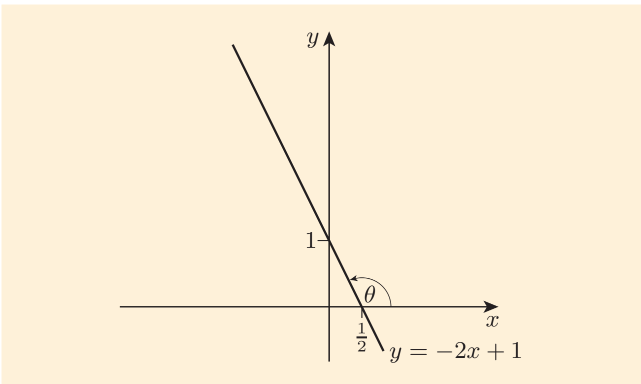
The angles in $\triangle ABC$ add up to 180°. So

$\gamma = 180° - 120° - 45° = 15°.$

That is, the acute angle between the two lines is 15°.

Activity 14

(a) A sketch of the line is shown below.



(b) The gradient of the line is -2, so

$$\tan\theta = -2.$$

The related acute (first-quadrant) angle is $\tan^{-1}(2) = 63^\circ$ (to the nearest degree).

The angle θ is an obtuse (second-quadrant) angle, so $\theta = 180^\circ - 63^\circ = 117^\circ$ (to the nearest degree).

That is, the angle of inclination is about 117°.

Activity 15

(a) The equation is

$$\cos\theta = \frac{\sqrt{3}}{2}.$$

The cosine of θ is positive, so θ is a first- or fourth-quadrant angle.

One solution is

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

The other solution is

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$$

(b) The equation is

$$\tan\theta = -\sqrt{3}.$$

The tangent of θ is negative, so θ is a second- or fourth-quadrant angle.

The related first-quadrant angle is

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

The solutions are

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3},$$

$$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

(c) The equation is

$$\cos\theta = 0.4.$$

The cosine of θ is positive, so θ is a first- or fourth-quadrant angle.

One solution is

$$\theta = \cos^{-1}(0.4) = 1.1592\ldots = 1.16 \text{ (to 3 s.f.)}.$$

The other solution is

$$\theta = 2\pi - 1.1592\ldots = 5.1239\ldots = 5.12 \text{ (to 3 s.f.)}.$$

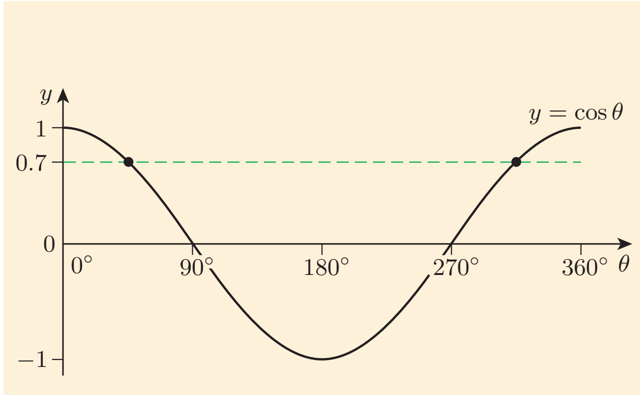
(You can check these solutions on your calculator.)

Activity 16

(a) The equation is

$$\cos\theta = 0.7.$$

A suitable sketch graph is as follows.



One solution is

$$\theta = \cos^{-1}(0.7) = 46^\circ \text{ (to the nearest degree)}.$$

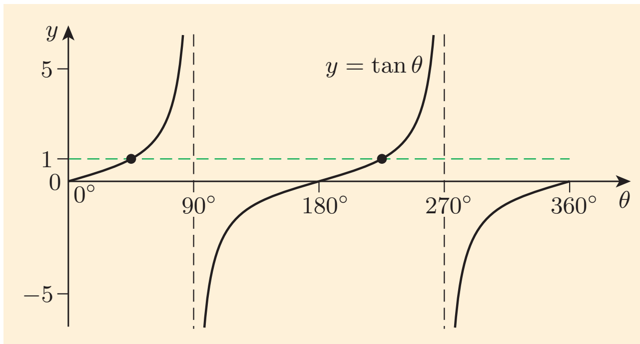
The other solution is as far below 360° as the first solution is above 0°, so it is

$$\theta = 360^\circ - 46^\circ = 314^\circ \text{ (to the nearest degree)}.$$

(b) The equation is

$$\tan\theta = 1.$$

A suitable sketch graph is as follows.



One solution is

$$\theta = \tan^{-1}(1) = 45^\circ.$$

The other solution is as far above 180° as the first solution is above 0°, so it is

$$\theta = 180^\circ + 45^\circ = 225^\circ.$$

(You can check these solutions on your calculator.)

Activity 17

(a) An expression for the total number of animals is $c + r$. Also, chickens have 2 feet and rabbits have 4 feet, so an expression for the total number of feet is $2c + 4r$.

So c and r satisfy the simultaneous equations

$$c + r = 50,$$

$$2c + 4r = 122.$$

Rearranging the first equation gives $r = 50 - c$, and substituting this into the second equation gives

$$2c + 4(50 - c) = 122$$
$$2c + 200 - 4c = 122$$
$$200 - 2c = 122$$
$$2c = 78.$$

Hence $c = 39$ and $r = 50 - 39 = 11$.

So there are 39 chickens and 11 rabbits.

(You might have solved the simultaneous equations by elimination rather than substitution.)

(b) The original solution to the problem as quoted in *Sacred mathematics* is as follows.

> If rabbits were chickens then the total number of feet would be 100, so we know the extra 22 feet are all from rabbits, which implies 11 rabbits and 39 chickens.

Activity 18

(a) There is no single answer because cats and dogs both have four feet, so equations involving numbers of feet do not distinguish between them. You might like to consider what happens if you write down the simultaneous equations for this problem. (Hint: See Unit 7, Activity 25.)

(b)–(d) No specific comments are provided for these question parts: be creative and remember that you can compare your answers with the numbers of animals that you started with.

Activity 19

(a) The area of the large square is $(a + b)^2$.

(b) The area of triangle WEH is $\frac{1}{2}ab$.

(c) The triangles XFE, YGF and ZHG also each have an area of $\frac{1}{2}ab$. So the total area of the four triangles is $2ab$. The area of the small square can be found by subtracting the areas of the triangles from the area of the large square. So the area of the small square is

$$(a + b)^2 - 2ab = a^2 + 2ab + b^2 - 2ab = a^2 + b^2.$$

(d) The area of the small square is also c^2. So

$$a^2 + b^2 = c^2.$$

Activity 20

(a) For example, consider $\frac{1}{3}$ and $\frac{2}{3}$. Squaring the larger and adding the smaller gives

$$\left(\tfrac{2}{3}\right)^2 + \tfrac{1}{3} = \tfrac{4}{9} + \tfrac{1}{3} = \tfrac{7}{9}.$$

Squaring the smaller and adding the larger gives

$$\left(\tfrac{1}{3}\right)^2 + \tfrac{2}{3} = \tfrac{1}{9} + \tfrac{2}{3} = \tfrac{7}{9}.$$

So the two answers are the same. This applies to whichever numbers you try (even if one is positive and one is negative – for example, 3 and -2).

So the conjecture is that the two sums are always the same.

(b) Since the numbers add up to 1, the smaller number is $1 - x$. So the first sum is

$$x^2 + (1 - x) = x^2 - x + 1$$

and the second sum is

$$(1 - x)^2 + x = x^2 - 2x + 1 + x = x^2 - x + 1.$$

The two sums are equal, so this proves the conjecture.

Activity 21

(a) AB is the side of the square, so it is 2 units long; BC is half of the side of the square, so it is 1 unit long. Since $\triangle ABC$ is right-angled, the cosine ratio can be used. This gives

$$\cos\angle ABC = \frac{\text{adj}}{\text{hyp}} = \tfrac{1}{2}.$$

So

$$\angle ABC = \cos^{-1}\left(\tfrac{1}{2}\right) = 60°.$$

Alternatively, $\triangle ABC$ is one half of an equilateral triangle, so $\angle ABC = 60°$.

(b) Since the paper has been folded over, α is half of $\angle ABC$. From part (a), $\angle ABC = 60°$, so

$$\alpha = 30°.$$

Also, as mentioned in the question, there are two layers of paper between the edges DB and EB, and an angle of a square is 90°, so

$$2(\alpha + \beta) = 90°.$$

Hence

$$\alpha + \beta = 45°,$$

and therefore

$$\beta = 45° - 30° = 15°.$$

The triangle that contains α and γ is right-angled. So

$$\gamma = 90° - 30° = 60°.$$

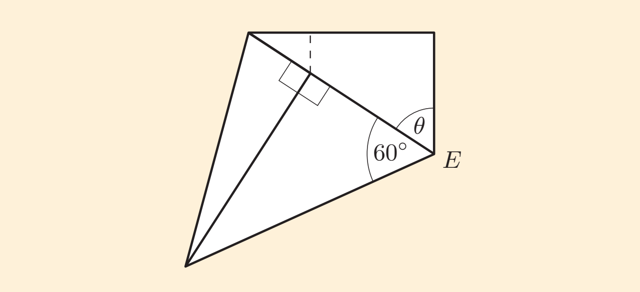
Similarly, the triangle that contains β and δ is right-angled. So

$$\delta = 90° - 15° = 75°.$$

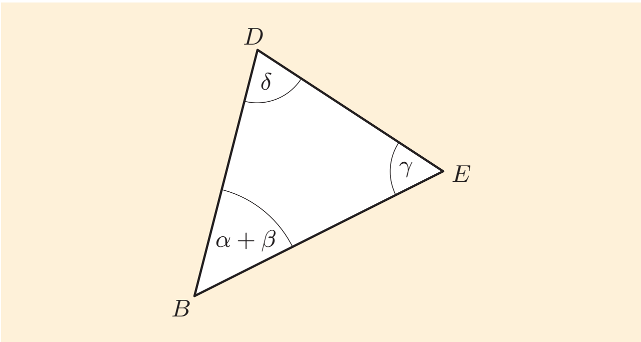
(c) The diagram at the top of the next page shows the folded square after step 3. An angle has been marked as θ. You can see that, if you were to unfold the paper again, then the three angles at E would be angles on a straight line. So

$$\theta = 180° - 2 \times 60° = 60°.$$

Hence, when the top right corner of the square is folded down in step 4, the angle θ lies exactly on top of the angle marked as 60°, and so the folded-down corner lies exactly on the edge of the paper shape.



(d) The three angles of the final triangle are $\alpha + \beta$, γ and δ, as shown below.



The sizes of these angles are as follows:

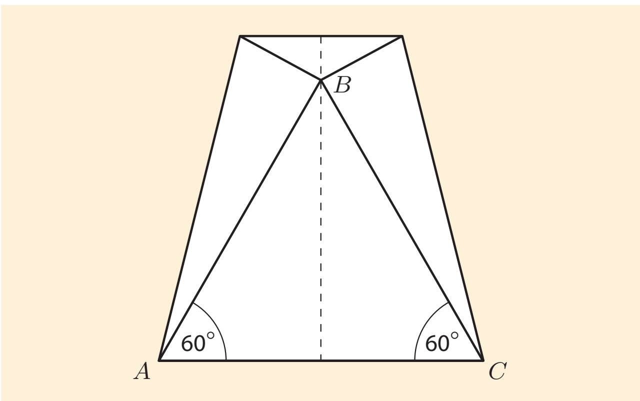
$$\alpha + \beta = 15° + 30° = 45°,$$
$$\gamma = 60°,$$
$$\delta = 75°.$$

(Check: $45° + 60° + 75° = 180°$.)

Activity 22

(a) The angles in an equilateral triangle are all 60°.

(b) One way to fold the square into an equilateral triangle is as follows. First fold the top left corner to the centre line, as in step 2 of Activity 21. Then fold the top right corner to the centre line, as shown in the diagram below. Each side of $\triangle ABC$ was originally a side of the square, so this triangle is equilateral. If the paper is now folded back along a horizontal line through B, and finally folded along the lines AB and BC, then the square is folded into an equilateral triangle.



(There are other solutions.)

Activity 23

(a) In the diagram, $h = OG + ON$. The distance OG is 32 metres, as before, and the distance ON can be found by using the right-angled triangle OPN, as before. This gives

$$\cos\phi = \frac{\text{adj}}{\text{hyp}} = \frac{ON}{30},$$

so

$$ON = 30\cos\phi.$$

Hence

$$h = OG + ON = 32 + 30\cos\phi.$$

(b) Since $\theta = \pi - \phi$, it follows from the related angles diagram that the two angles θ and ϕ have the same cosine values, except possibly for the signs. Since ϕ and θ are first- and second-quadrant angles, respectively, it follows from the CAST diagram that their cosines are positive and negative, respectively. Hence $\cos\theta = -\cos\phi$.

(c) The equation $\cos\theta = -\cos\phi$ is equivalent to $\cos\phi = -\cos\theta$. Using this equation to substitute in the equation found in part (a) gives

$$h = 32 + 30(-\cos\theta) = 32 - 30\cos\theta.$$

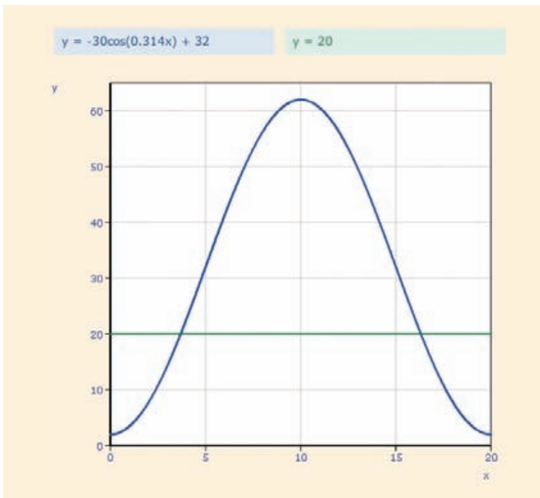
This is the formula that was found for acute angles θ.

Activity 24

(a) The correct values for the constants in the first equation are $a = -30$, $b = 0.314$, $c = 0$ and $d = 32$, since these values give the graph of the equation $y = -30\cos(0.314(x - 0)) + 32$, which is equivalent to $y = 32 - 30\cos(0.314x)$. The correct values for the constants in the second equation are $m = 0$ and $c = 20$.

Suitable values for the minimum and maximum values on the y-axis are 0 and 65, respectively (since the height in metres of a gondola is always between these numbers).

The graph obtained is shown below.



(b) The line and the curve cross when $x = 4$ and $x = 16$ (both to the nearest whole number). So the best views become visible approximately 4 minutes after the gondola was at its lowest point, and cease to be visible approximately 16 minutes after the gondola was at its lowest point.

Activity 25

(a) Changing the value of a stretches or compresses the graph in the y-direction. The maximum value reached by y is the magnitude of a (for example, if $a = 2$, then the maximum value is 2, and if $a = -2$, then the maximum value is also 2). The minimum value reached by y is the negative of the magnitude of a.

If you change the value of a to its negative, then the graph is reflected in the x-axis. So in the oscillation that starts when $x = 0$, the value of y first decreases to the minimum value and then increases to the maximum value, instead of the other way round.

(b) You should have found that if b is a positive integer, then the number of complete oscillations in the interval from 0 to 2π is b. This result also holds if b is fractional: for example, if $b = 2.5$, then there are two and a half oscillations in the interval from 0 to 2π.

If b is negative, then the number of oscillations in the interval from 0 to 2π is the magnitude of b.

If you change the value of b to its negative, then the graph is reflected in the y-axis. So, as in part (a), in the oscillation that starts when $x = 0$, the value of y first decreases to the minimum value and then increases to the maximum value, instead of the other way round.

(c) If you start with $c = 0$ and change the value of c, then the graph moves along the x-axis by c units. For example, if $c = 2$, then the graph is the same shape as the graph of $y = \sin x$, but moved a distance of 2 to the right. Similarly, if $c = -2$, then the graph is the same shape as the graph of $y = \sin x$, but moved a distance of 2 to the left.

So, in general, the graph of of $y = \sin(x - c)$ seems to be the same shape as the graph of $y = \sin x$, but moved to the right by distance c (the move is to the left if c is negative).

(d) As d changes, the graph moves up or down the y-axis. For example, if $d = 1$, then the graph is the same shape as the graph of $y = \sin x$, but moved 1 unit up the y-axis.

(e) The graph of the general cosine function changes in the same way as the graph of the general sine function when the values of the constants a, b, c and d are changed. This is not surprising, since the sine and cosine functions have the same basic shape. In fact, you can obtain the graph of $y = \cos x$ by shifting the graph of $y = \sin x$ by $\frac{\pi}{2}$ radians to the left, that is, by drawing the graph of $y = \sin\left(x + \frac{\pi}{2}\right)$.

Activity 26

Each of the equations is of the form

$$y = a\sin(b(x - c)) + d$$

or

$$y = a\cos(b(x - c)) + d.$$

(a) Here $b = 4$, so the period is $2\pi/4 = \pi/2$.

(b) Here $b = 5$, so the period is $2\pi/5$.

(c) Here $b = \frac{1}{2}$, so the period is $2\pi/0.5 = 4\pi$.

(d) Here $b = -1$, so the period is $2\pi/1 = 2\pi$.

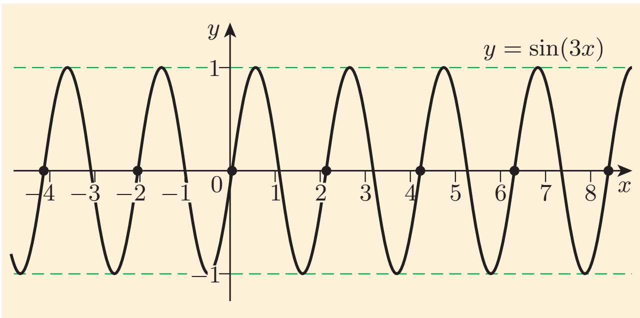
(You might like to check these answers by plotting the graphs on Graphplotter.)

Activity 27

(a) The equation is $y = \sin 3x$. Here $a = 1$, $b = 3$, $c = 0$ and $d = 0$. So

amplitude $= 1$,
period $= 2\pi/3 \approx 2.09$,
horizontal displacement $= 0$,
vertical displacement $= 0$,
minimum value $= -1$,
maximum value $= 1$.

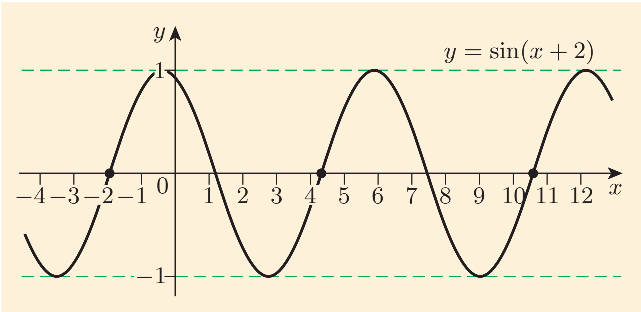
There are three complete oscillations in an interval of 2π. The graph is shown below.



(b) The equation is $y = \sin(x + 2)$. Here $a = 1$, $b = 1$, $c = -2$ and $d = 0$. So

amplitude $= 1$,
period $= 2\pi/1 = 2\pi \approx 6.28$,
horizontal displacement $= -2$,
vertical displacement $= 0$,
minimum value $= -1$,
maximum value $= 1$.

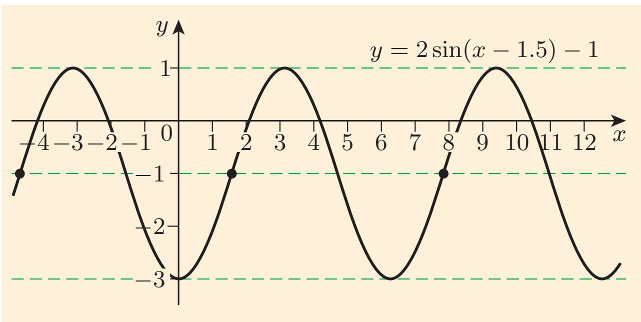
The graph is obtained by shifting the graph of $y = \sin x$ by 2 to the left, as shown below.



(c) The equation is $y = 2\sin(x - 1.5) - 1$. Here $a = 2$, $b = 1$, $c = 1.5$ and $d = -1$. So

amplitude $= 2$,
period $= 2\pi/1 = 2\pi \approx 6.28$,
horizontal displacement $= 1.5$,
vertical displacement $= -1$,
minimum value $= -1 - 2 = -3$,
maximum value $= -1 + 2 = 1$.

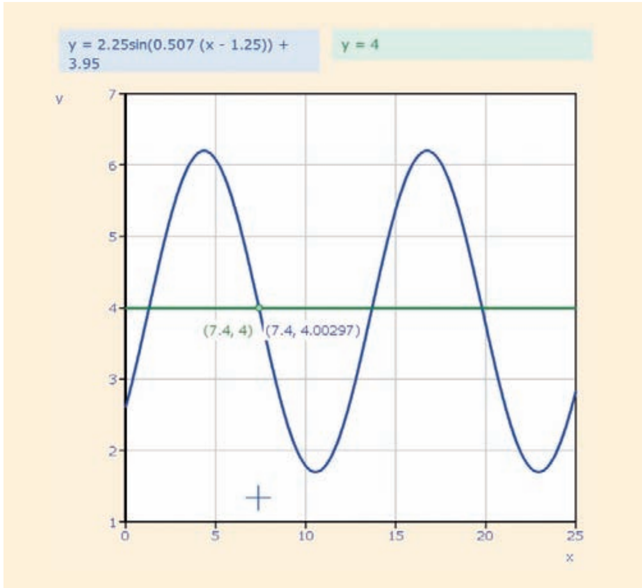
The graph is shown below.



Activity 28

(a) The y-coordinate corresponding to the x-coordinate 4.35 is 6.2, and this agrees with the height of the high tide, which is 6.2 metres.

(b) If you add the line $y = 4$ to the graph (by using 'Two graphs'), then you can read off the x-coordinates of the crossing points, as illustrated below.



According to the model, the tide is 4 metres or higher between approximately 1.3 hours and 7.4 hours, and again between 13.7 hours and 19.8 hours.

(In practice, answers in hours and minutes may be needed. To obtain such answers, you should first find them in hours to more decimal places than given above, and then convert them to hours and minutes.)